

ON MAXIMUM PRINCIPLES FOR EQUATIONS OF MIXED ELLIPTIC- HYPERBOLIC TYPE OF THE SECOND KIND

M. E. Lerner

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Abstract

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MATHEMATICS

M. E. LERNER

ON MAXIMUM PRINCIPLES FOR EQUATIONS OF MIXED ELLIPTIC-HYPERBOLIC TYPE OF THE SECOND KIND

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Consider the equation

$$K_m[u] \equiv u_{xx} + \operatorname{sgn} y \cdot |y|^m u_{yy} + \frac{M(x; y)}{|y|^\beta} u_x + \frac{N(x; y)}{|y|^\alpha} u_y + \frac{F(x; y)}{|y|^\gamma} u = 0 \quad (K_m)$$

in a domain D , bounded by: 1) a simple Jordan arc σ , situated in the upper half-plane and resting on the axis $y = 0$ at the points $O(0; 0)$, $A(a, 0)$, $a > 0$; 2) the characteristics OC and AC . Let D_1 and D_2 be respectively the subdomains of ellipticity and hyperbolicity of the equation (K_m) , OA its line of transition; $0 < m < 2$; $M, N, F \in C^{(0)}(\bar{D})$; $M, N \in C^{(1)}(\bar{D}_2)$; $F \leq 0$ in D_1 . Particular cases of (K_m) are the equations studied in a mixed domain

$$u_{xx} + \operatorname{sgn} y \cdot |y|^m u_{yy} = 0, \quad 0 < m < 1 \quad (1); \quad (\Sigma_m)$$

$$u_{xx} + y u_{yy} + a u_y = 0, \quad a < 1 \quad (2-5); \quad (S_1^\alpha)$$

$$u_{xx} + \operatorname{sgn} y \cdot |y|^m u_{yy} + a |y|^{m-1} u_y = 0, \quad 0 < m < 2, \quad m/2 < \alpha < 1 \quad (6). \quad (S_m^\alpha)$$

We shall say that for equation (K_m) in the domain D (in the domain D_2), in a certain class $[K]$ of its solutions, maximum principle I (maximum principle II) holds in the following case: let $u \in [K]$ and $u|_{OC} \equiv 0$; then $\max_{\bar{D}} |u|$ ($\max |u|$) is attained on $\sigma \cup AC$ (on the characteristic AC).

§ 1. Consider the equation

$$E[u] \equiv u_{\xi\eta} + a(\xi; \eta) u_\xi + b(\xi; \eta) u_\eta + c(\xi; \eta) u = 0 \quad (E)$$

in the characteristic triangle $O_0C_0A_0$ (the domain Δ , $O_0(0;0)$, $C_0(0;-\xi_0)$, $A_0(\xi_0;-\xi_0)$, $\xi_0 > 0$); $a, a_\xi, b, c \in C^{(0)}(\Delta \cup O_0C_0)$.

We shall say that in the domain Δ the coefficients of equation (E) satisfy conditions (L), if: 1) $a > 0$ on O_0C_0 ; 2)

$$a(P) > \int_{PQ} \{2|\gamma_2| + \beta|c_2|\} d\xi$$

in Δ , and conditions (M), if in $\Delta \cup O_0C_0$: 1) $a > Q$; 2)

$$a(Q)\beta(Q) > \int_{PQ} |\gamma| d\xi.$$

Here PQ is a segment of an arbitrary characteristic $\eta = \text{const}$,

$$P \in O_0C_0, \quad Q \in \Delta; \quad \beta = \exp \left\{ \int_0^\xi b d\xi \right\}; \quad h = a_\xi + ab - c = h_1 + h_2, \quad h_1 \geq 0 \text{ in } \Delta;$$

$$\gamma = \gamma_1 + \gamma_2, \quad \gamma_1 = -\beta h_1, \quad \gamma_2 = -\beta h_2; \quad c = c_1 + c_2, \quad c_1 \geq 0.$$

We shall call the function $u(\xi; \eta)$ a solution of equation (E) of class $[W_0]$, if: $E[u] \equiv 0$ in Δ , $u \in C^{(2)}(\Delta)$, $u \in C^{(0)}(\overline{\Delta})$, $u \in C^{(1)}(\Delta \cup O_0C_0)$.

Lemma 1 (the characteristic principle of the absolute extremum for hyperbolic equations). Let $u(\xi; \eta)$ be a solution of equation (E) of class $[W_0]$, $u \neq 0$ in Δ , and $u|_{O_0C_0} = 0$. Then $\max_\Delta |u|$ cannot be attained in Δ , but is attained on the characteristic $\overline{A_0C_0}$.

1. Suppose the contrary. Let $\max_\Delta |u|$ be attained at some point $Q \in \Delta$. We write equation (E) in the form

$$(\beta u_\eta)_\xi + (\alpha \beta u)_\xi + \gamma u = 0$$

and integrate this equation over the segment PQ ($P \in O_0C_0$, $\eta_P = \eta_Q$). We obtain

$$\begin{aligned} \beta(Q)u_\eta(Q) = & \left[-a(P)u(Q) - \int_{PQ} [u(Q) - u]\gamma_2 d\xi - u(Q) \int_{PQ} \beta C_2 d\xi \right] + \\ & + \int_{PQ} [u(Q) - u]\gamma_1 d\xi - u(Q) \int_{PQ} \beta C_1 d\xi; \end{aligned} \quad (1)$$

$$\beta(Q)u_\eta(Q) = -\beta(Q)a(Q)u(Q) - \int_{PQ} \gamma u d\xi. \quad (2)$$

By virtue of conditions (L) and (M), respectively, from (1) and (2) it follows that

$$u(Q)u_\eta(Q) < 0, \quad (3)$$

which is impossible.

2. Suppose that $\max_\Delta |u|$ is not attained on $\overline{A_0C_0}$. Then, by the preceding part, it is attained only on the open segment O_0A_0 at some point Q . Consequently, in Δ , in some neighborhood of the point Q , there exists a point Q' at which $u \neq 0$ and

$$|u(Q')| > \max_{A_0C_0} |u|. \quad (4)$$

Draw through the point Q a segment $O'_0A'_0 \parallel O_0A_0$ ($O'_0 \in O_0C_0$, $A'_0 \in C_0A_0$), and denote by Δ' the open triangular domain $O'_0C_0A'_0$. $\max_{\Delta'} |u|$ cannot be attained at a point belonging to $\Delta' \cup O'_0A'_0$, since, by the preceding part, at such a point one would have $u \cdot u_\eta < 0$. Hence $\max_{\Delta'} |u|$ is attained only on $\overline{A'_0C_0}$, and

$$|u(Q')| \leq \max_{A'_0C_0} |u|,$$

which contradicts (4) and completes the proof of the lemma.

It can be shown that $\max_\Delta |u|$ is attained only on the characteristic A_0C_0 , if

$$|u(A)| < \max_{A_0C_0} |u|.$$

Denote by $(E_m^{\lambda\mu})$ equation (E) in which

$$\begin{aligned} a &= -\sigma/2(\xi + \eta) + f(\xi; \eta)/|\xi + \eta|^{\sigma+\lambda}, & b &= -\sigma/2(\xi + \eta) + g(\xi; \eta)/|\xi + \eta|^{\sigma+\lambda}, \\ c &= r(\xi; \eta)/|\xi + \eta|^\mu, & \sigma &= m/(2-m), \quad 0 < m < 1, \quad \lambda < 2(1-m)/(2-m), \\ & & \mu &< (4-3m)/2(2-m); \quad f, (\xi + \eta)f_\xi, g, r \in C^{(0)}(\overline{\Delta}). \end{aligned}$$

Lemma 2. For equation $(E_m^{\lambda\mu})$, the assertion of Lemma 1 is valid if the length of the segment O_0A_0 is sufficiently small.

The proof follows from the fact that conditions (L) are satisfied if the length of the segment O_0A_0 is sufficiently small.

We note that, under the change of variables

$$\xi = x - \frac{2}{2-m}(-y)^{(2-m)/2}, \quad \eta = -x - \frac{2}{2-m}(-y)^{(2-m)/2},$$

equation (K_m) is transformed into an equation of the form (E), and the triangle OAC into a triangle of type Δ .

§ 2. We shall call a function $u(x; y)$ a solution of equation (K_m) of class $[W]$ if:

$$K_m[u] \equiv 0 \quad \text{in } D_1 \cup D_2, \quad u \in C^{(2)}(D_1 \cup D_2), \quad u \in C^{(0)}(\overline{D}), \\ u \in C^{(1)}(\overline{D} \setminus \overline{OA}).$$

Theorem 1. For equation (K_m) in the domain D (in the domain D_2), in the class $[W]$ of its solutions there holds the maximum principle I (the maximum principle—

II), if the length of the transition line is sufficiently small and $0 < m < 1$, $\beta < 1 - m/2$, $\alpha < 1 - m$, $\gamma < 1 - 3m/4$.

Proof. Let $u(x; y)$ be an arbitrary solution of equation (K_m) of class $[W]$, $u|_{\overline{OC}} \equiv 0$ and $u \not\equiv 0$ in D . Then, by the sufficient smallness of the transition line, by Lemma 2, $\max_{\overline{D_2}} |u|$ is attained on $\overline{AC}$, and maximum principle II is proved.

Let $\max_{\overline{D}} |u|$ be attained in $\overline{D_2}$. Consequently, it is attained on $\overline{AC}$.

Let $\max_{\overline{D}} |u|$ be attained in $\overline{D_1}$. Then, by the known property of elliptic equations, it is attained on $\overline{\sigma} \cup OA$, and, if on OA , then, by the preceding, also on AC . Thus, $\max_{\overline{D}} |u|$ is attained on $\overline{\sigma} \cup AC$, and the theorem is completely proved.

Theorem 2. For equation (K_m) in the domain D (in the domain D_2), in the class $[W]$ of its solutions, maximum principle I (maximum principle (II)) is valid if its coefficients are such that the coefficients of the corresponding equation of the form (E) satisfy in Δ the conditions (I) or (M).

The proof is analogous to Theorem 1 and follows from Lemma 1.

We note that in Theorem 2 the requirement of continuity on the characteristic AC of the coefficients of equation (K_m) and of the first partial derivatives of the functions $M(x; y)$ and $N(x; y)$ is superfluous.

Corollary of Theorem 2. For an arbitrary length of the transition line, for equation (Σ_m) with $0 < m < 2$ and for equation (S_m^α) with $0 < m < 2$, $\alpha < m/2$, in the class $[W]$ maximum principles I and II are valid.

§ 3. Suppose a one-to-one correspondence has been established between all points of the characteristic $\overline{AC}$ and some set of points $E \subset \sigma$; Q_1 and Q_2 are arbitrary corresponding points ($Q_1 \in E$, $Q_2 \in \overline{AC}$), $E_1 \equiv \overline{\sigma} \setminus E$, and at each point belonging to the set E , the arc σ has a normal.

Problem Φ . Find a solution $u(x; y)$ of equation (K_m) of class $[W]$ from the data:

$$1. u|_{E_1} = \varphi_1. \quad 2. \partial u / \partial n|_E = \varphi_2. \quad 3. u|_{\overline{OC}} = \psi. \quad 4. u(Q_1) - u(Q_2) = g(Q_1).$$

Here $\varphi_1, \varphi_2, \psi, g$ are functions continuous on the indicated parts of the boundary of the domain D ; n is the interior normal. Boundary condition 4 defines a "curvilinear jump of compaction" ^(7,8).

Theorem 3. For equation (K_m) , under the conditions of Theorems 1 and 2, the solution of problem Φ is unique.

Suppose the contrary. Let $u(x; y)$ be a solution of problem Φ for equation (K_m) , satisfying homogeneous boundary conditions, but $u \not\equiv 0$ in D . By homogeneous boundary condition 3, $\max_{\bar{D}} |u|$ is attained on $\bar{\sigma} \cup AC$. Taking into account homogeneous boundary conditions 1 and 2, we obtain that $\max_{\bar{D}} |u|$ cannot be attained on $\bar{\sigma}$ and, consequently, is attained on AC . But then, by homogeneous boundary condition 4, it is attained on $\bar{\sigma}$, which, as has already been shown, is impossible. The contradiction obtained proves the theorem.

Corollary of Theorem 3. For equations (Σ_m) and (S_m^α) , under the conditions of the corollary to Theorem 2, the solution of problem Φ is unique.

We note that for equation (S_1^α) , when $\alpha < 0$, there exists and is unique a solution of the Dirichlet problem in the class of functions continuous in $\bar{D}$ and having a continuous derivative with respect to y on the transition line ⁽⁴⁾.

Remark 1. We have considered maximum principles I and II with data on the characteristic $\bar{OA}$. These principles are formulated analogously with data on the characteristic AC . For their proof, in equation (E) one should interchange the roles of the variables ξ and η . Maximum principles I and II with data on the characteristic AC hold for equation (K_m) under the conditions of Theorem 1 and for equations (Σ_m) and (S_m^α) under the conditions of Theorem 2.

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Kuibyshev Polytechnic Institute
named after V. V. Kuibyshev

Kuibyshev State Pedagogical Institute
named after V. V. Kuibyshev

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