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Abstract

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MATHEMATICS

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HOMOLOGY K -GROUPS AS GENERALIZED STEENROD GROUPS

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In the preceding note ⁽¹⁾ we introduced generalized Steenrod homology groups, generalizing the classical construction of Steenrod set forth in ⁽²⁾. Here we shall show that the homology K -groups introduced by the author in ⁽³⁾ also fit into the scheme of generalized Steenrod homology groups. From this, in particular, it follows that the homology groups considered by K. A. Sitnikov in the paper ⁽⁴⁾ are also a special case of generalized Steenrod groups.

By K we shall denote a fixed locally finite complex. Further, if L is an arbitrary locally finite complex, then by L^* we shall denote the barycentric subdivision of the cellular complex $K \times L$, i.e.

$$L^* = (K \times L)'.$$

Let, further, L and M be locally finite complexes and let $f : L \rightarrow M$ be some closed simplicial embedding (see ⁽¹⁾). Then the map

$$1_K \times f : K \times L \rightarrow K \times M,$$

which is cellular, is defined. If we take the barycentric subdivisions of the complexes $K \times L$ and $K \times M$, then we obtain simplicial complexes $L^* = (K \times L)'$ and $M^* = (K \times M)'$, and the map $1_K \times f$ will already be a simplicial map of the first complex into the second. This simplicial map (which, as is easy to see, is a closed simplicial embedding) will be denoted by f^* . Thus, every closed simplicial embedding $f : L \rightarrow M$ gives rise to a new closed simplicial embedding $f^* : L^* \rightarrow M^*$. The category consisting of all simplicial complexes of the form L^* and their closed simplicial embeddings of the form f^* will be denoted by $\mathfrak{A}$. It is easily verified that conditions 1 and 2 imposed in note ⁽¹⁾ on the category $\mathfrak{A}$ are satisfied here.

We now define, for each complex L^* that is an object of the category $\mathfrak{A}$, a certain class of chains $\alpha_p(L^*)$, $p = 0, 1, 2, \dots$. Namely, let u_{p+r} be an arbitrary (infinite) $(p+r)$ -dimensional chain of the locally finite complex L over the group G , and let τ^r be an arbitrary r -dimensional simplex of the complex K ; then the barycentric subdivision $(u_{p+r}, \tau^r)^*$ of the chain $u_{p+r} \times \tau^r$ of the locally finite complex $K \times L$ is a chain of the complex $L^* = (K \times L)'$. We shall include all possible chains of the form

$$\sum_{\substack{\tau^r \in K \\ r=0,1,2,\dots}} (u_{p+r}, \tau^r)^*$$

in $\alpha_p(L^*)$.

We define the boundary operator $d : \alpha_p(L^*) \rightarrow \alpha_{p-1}(L^*)$ in accordance with the boundary operator in $(^3)$. Namely, we set

$$d(u_{p+r}, \tau^r)^* = (\partial u_{p+r}, \tau^r)^* + (-1)^{p+r+1} (u_{p+r}, \delta \tau^r)^*, \quad (1)$$

where ∂ is the boundary operator in L , and δ is the coboundary operator in K . From this formula it follows easily that $d \circ d = 0$. Further, it is easy to see that the formula

(1) can be rewritten in the following equivalent form:

$$\begin{aligned} d \left(\sum_{\tau} (x_{\tau}, \tau)^* \right) &= \sum_{\tau} (\partial(x_{\tau}), \tau)^* - \sum_{\tau} \sum_{\sigma} (-1)^{\dim x_{\sigma}} [\tau : \sigma] (x_{\sigma}, \tau)^* \\ &= \sum_{\tau} (\partial(x_{\tau}), \tau)^* + \sum_{\tau} (-1)^{\dim x_{\tau}} (x_{\partial \tau}, \tau)^*, \end{aligned} \quad (2)$$

where τ ranges over all (arbitrarily oriented) cells of the complex K ; further, σ ranges over cells related to τ by the relation $\dim \sigma = \dim \tau - 1$, and $[\tau : \sigma]$ denotes the incidence coefficient of the cells τ and σ in the complex K . Formula (2) shows that the operator d corresponds to the boundary operator defined in note $(^3)$. More precisely, let $x = \{x_{\tau}\}$ be a collection of chains of the complex L , indexed by the cells $\tau \in K$, such that $\dim x_{\tau} = p + \dim \tau$, where p is a fixed nonnegative integer. In other words, x is a p -dimensional K -chain of the complex L in the sense of note $(^3)$. By $\partial x = \{(\partial x)_{\tau}\}$ we denote the boundary of this K -chain x (in the sense of $(^3)$). Then, as formula (2) shows,

$$d \left(\sum_{\tau} (x_{\tau}, \tau)^* \right) = \sum_{\tau} ((\partial x)_{\tau}, \tau)^*, \quad (3)$$

which establishes the correspondence between the operator

$$d : \alpha_p(L^*) \rightarrow \alpha_{p-1}(L^*)$$

and the operator

$$\partial : C_p^K(L, G) \rightarrow C_{p-1}^K(L, G).$$

Thus we have a category $\mathfrak{A}$, consisting of simplicial complexes of a certain kind (namely, complexes of the form $L^* = (K \times L)'$) and their closed simplicial embeddings of a certain kind (namely, maps of the form f^*). Further, for every complex L^* that is an object of the category $\mathfrak{A}$, we have a certain class of chains $\alpha(L^*)$ (and, as is easy to check, the conditions 1-4 imposed on the class of chains in note ⁽¹⁾ are satisfied here). Therefore we can define the $(\mathfrak{A}, \alpha)$ -homology groups of an arbitrary compactum Φ , as indicated in note ⁽¹⁾.

Theorem. The homology groups constructed here

$$H_p^{(\mathfrak{A}, \alpha)}(\Phi, G)$$

are naturally isomorphic to the K -homology groups of the compactum Φ over the coefficient group G :

$$H_p^{(\mathfrak{A}, \alpha)}(\Phi, G) \simeq \Delta_p^K(\Phi, G).$$

The proof of this theorem is carried out according to the following scheme. Let $L^* = (K \times L)'$ be a complex belonging to the category $\mathfrak{A}$, and let $\varphi : (L^*)^0 \rightarrow \Phi$ be a regular mapping of its zero-dimensional skeleton into the metric compact space Φ . Further, let $u \in \alpha_p(L^*)$, so that the triple (L^*, φ, u) is a p -dimensional regular chain of the space Φ over the coefficient group G . The chain u can be written in the form

$$u = \sum_{\tau \in K} (u_\tau, \tau)^*,$$

where u_τ is some chain of the complex L , with $\dim u_\tau = p + \dim \tau$. Now let $\varphi_\tau : L^0 \rightarrow \Phi$ be the mapping of the zero-dimensional skeleton of the complex L into Φ , defined by the relation

$$\varphi_\tau(a) = \varphi(o_\tau \times a), \quad a \in L^0,$$

where o_τ is the center of the simplex τ . The mapping φ_τ carries each simplex of L into some skeleton of the space Φ . As a result, the chain u_τ passes into

some chain of nerves of the space Φ , which we shall denote by x_τ . It is readily verified that the degrees of fineness of the chains x_τ decrease without bound

when τ “goes to infinity” in the complex K , i.e., $x = \{x_\tau\}$ is a p -dimensional K -chain of the space Φ over the group G . The correspondence thus constructed

$$(L^*, \varphi, u) \rightarrow \{x_\tau\}$$

between regular p -dimensional chains of the space Φ and p -dimensional K -chains of the space Φ is precisely what establishes the isomorphism indicated in the theorem. Formulas (1), (2), (3) give the algebraic basis for verifying that this mapping is an isomorphism.

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Note: Figure translations are in progress. See original paper for figures.

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