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Abstract

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MATHEMATICAL PHYSICS

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TRACE FORMULA FOR A SYSTEM OF THREE PARTICLES

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1. Notation. Let $R(z)$ be the resolvent of the quantum-mechanical energy operator H of a system of three pairwise interacting nonrelativistic particles with the motion of the center of inertia separated out. Let $R_0(z)$ be the resolvent of the kinetic-energy operator H_0 , and let $R_\alpha(z)$ ($\alpha = 1, 2, 3$) be the resolvent of the operator H_α , in which only the interaction of the α -th pair of particles is taken into account. All these operators act in the Hilbert space $L_2(R^6)$. A more detailed description of them may be found in the work of L. D. Faddeev ⁽¹⁾, the results of which are used essentially in the present note.

Suppose that the Fourier transforms $v_\alpha(k)$, $k \in R^3$, of the potentials of the pair interactions satisfy the conditions: 1) $v_\alpha(-k) = v_\alpha(k)$; 2) there exist continuous derivatives $D^\chi v_\alpha$ with respect to k of order $|\chi|$, $0 \leq |\chi| \leq 5$; 3) $|(D^\chi v_\alpha)(k)| \leq C(1 + |k|)^{-1-\theta}$, $\theta > 1/2$; 4) the energy operators h_α of the α -th pair, with the motion of its center of inertia separated out, have only a finite number of negative eigenvalues $-\chi_{\alpha,i}^2$, $-\chi_{\alpha,i}^2 \in \mathfrak{S}_\alpha$; 5) the point $z = 0$ is not an exceptional point of certain special integral operators $a_\alpha(z)$ (see ⁽¹⁾, p. 26).

The connected part of the resolvent $R(z)$ is the operator

$$\Re(z) = R(z) - R_0(z) - \sum_{\alpha} (R_\alpha(z) - R_0(z)), \quad (1)$$

defined outside the spectrum Σ of the operator H .

Lemma 1. The operator

$$\Re_I(z) = \Re(z) - \Re(\bar{z}), \quad z \in \Sigma, \quad (2)$$

is a nuclear operator.

Let $\Phi = \mathfrak{S}_1 \cup \mathfrak{S}_2 \cup \mathfrak{S}_3 \cup \Sigma_0$, where Σ_0 is the set of eigenvalues of the operator H .

Lemma 2. Let $\lambda \in \Phi$; then the limit

$$\Omega_{\pm}(\lambda) = \lim_{\varepsilon \rightarrow \pm 0} \text{Sp } \Re_I(\lambda + i\varepsilon) \quad (3)$$

exists.

In ⁽²⁾, for the analogous expression $\omega(\lambda)$ in the case of an operator of type h (the energy of a pair), the trace formula

$$\omega_{\pm}(\lambda) = \frac{d}{d\lambda} \text{Sp} \ln s(\lambda) = \text{Sp} s^*(\lambda) \frac{ds(\lambda)}{d\lambda}, \quad \lambda > 0, \quad (4)$$

was obtained, where $s(\lambda)$ is the scattering matrix for the operator h . A similar formula is proved below for $\Omega(\lambda)$.

The trace formula for $\Omega(\lambda)$ was studied within the framework of the formal perturbation theory of F. A. Berezin ⁽³⁾. It was assumed there that $\Phi = \emptyset$, so that the scattering was considered one-channel. In the present work the trace formula is derived without this restriction; moreover, in the case of one-channel scattering, our results differ somewhat in notation from F. A. Berezin's formula, and we have not succeeded in verifying their identity.

2. Scattering matrix.

Consider the restriction of the operator H to the absolutely continuous subspace $\mathfrak{H}$. The usual direct-integral decomposition corresponding to it has the form

$$\mathfrak{H} \leftrightarrow \int_{-\chi^2}^{\infty} \oplus \mathfrak{H}(\lambda) d\lambda, \quad -\chi^2 = \min(-\chi_{\alpha,i}^2). \quad (5)$$

The arrow $\leftrightarrow$ denotes unitary equivalence. For $\lambda < 0$, $\lambda \in \Phi$,

$$\mathfrak{H}(\lambda) = \sum_{-\chi_{\alpha,i}^2 < \lambda} \oplus \mathfrak{H}_{\alpha,i}(\lambda), \quad \mathfrak{H}_{\alpha,i}(\lambda) = L_2(S_{\alpha}^2, \rho_{\alpha}(\lambda + \chi_{\alpha,i}^2) d\omega_{\alpha}). \quad (6)$$

Here $L_2(S_{\alpha}^2, \rho_{\alpha}(\lambda + \chi_{\alpha,i}^2) d\omega_{\alpha})$ is the space of functions on $S_{\alpha}^2 = \{\omega = (2n_{\alpha})^{1/2} p/|p| \mid p \in R^3\}$, square-summable with respect to the measure $\rho_{\alpha}(\lambda + \chi_{\alpha,i}^2) d\omega_{\alpha}$, $\lambda = p^2/2n_{\alpha} - \chi_{\alpha,i}^2$; $d\omega_{\alpha}$ is the surface element on S_{α}^2 , and $\rho_{\alpha}(\lambda + \chi_{\alpha,i}^2) d\omega_{\alpha} d\lambda = d^3p$. The constants denoted by the letters $m, n, m_{\alpha}, n_{\alpha}$ are various "reduced masses" described in (1). For $\lambda > 0$, in (5) the direct sum (6) is supplemented by the term

$$\mathfrak{H}_0(\lambda) = L_2(S_0^5, \rho_0(\lambda) d\omega_0), \quad (7)$$

where

$$S_0^5 = \{\omega = (k^2/2m + p^2/2n)^{-1/2} P \mid P = \{k, p\}; k, p \in R^3\}; \quad \lambda = k^2/2m + p^2/2n;$$

$d\omega_0$ is the surface element on S_0^5 , and $\rho_0(\lambda) d\omega_0 d\lambda = d^3k d^3p$.

The scattering operator $S : \mathfrak{H} \rightarrow \mathfrak{H}$ is given in the representation (5) by a family of unitary operators—the scattering matrix $S(\lambda) : \mathfrak{H}(\lambda) \rightarrow \mathfrak{H}(\lambda)$. The operators $S(\lambda)$ can be explicitly described by matrix kernels. In particular, for $\lambda < 0$ the kernels define the operator $S(\lambda)$ by the formula

$$g_{\alpha,i}(\omega_\alpha) = \sum_{\beta,k} \int_{S_\beta^2} S_{\alpha\beta}^{i,k}(\omega_\alpha, \omega'_\beta; \lambda) f_{\beta,k}(\omega'_\beta) d\omega'_\beta, \quad (8)$$

where $g = S(\lambda)f$, $g = \{g_{\alpha,i}(\omega_\alpha)\}$, $f = \{f_{\alpha,i}(\omega_\alpha)\}$. The kernels describing the action of $S(\lambda)$ on $f \in \mathfrak{H}_0(\lambda)$, $\lambda > 0$, are described analogously. The kernels defining $S(\lambda) - I$, I being the identity transformations of $\mathfrak{H}(\lambda)$, for $\lambda < 0$ are smooth functions of their arguments, while for $\lambda > 0$, generally speaking, they are singular generalized functions. The expression $dS(\lambda)/d\lambda$ will be characterized conventionally by the matrix kernels obtained by differentiating the kernels of the operator $S(\lambda)$ with respect to the variable λ .

3. Preparatory formulas.

With the aid of Hilbert's identity the following can be proved.

Basic preparatory formula. If $z = \lambda + i\mu$, $z \in \Sigma$, then the relation holds

$$\text{Sp } \mathfrak{R}_I(z) = \text{Sp} \left\{ A^*(z) \frac{dA(z)}{d\lambda} - \sum_{\alpha} A_{\alpha}^*(z) \frac{dA_{\alpha}(z)}{d\lambda} \right\}, \quad (9)$$

where $A(z) = E - 2i\mu R_0(z)T(z)R_0(\bar{z})$ is a unitary operator, and $T(z)$ is related to the resolvent by the relation

$$R(z) = R_0(z) - R_0(z)T(z)R_0(z).$$

At the same time $A_{\alpha}(z)$ is analogously related to the operators $T_{\alpha}(z)$ and $R_{\alpha}(z)$.

We shall carry out the further transformations of this formula in the momentum representation (see (1)).

By integration by parts in (9), two relations are established. **Preparatory formula for the senior channel:**

$$\text{Sp } \mathfrak{R}_I(z) = \text{Sp} \left\{ \mathcal{P}[T(z)] - \sum_{\alpha} \mathcal{P}[T_{\alpha}(z)] \right\}. \quad (10)$$

where

$$\mathcal{F}[A(z)] = -2i\mu R_0(z)R_0(\bar{z})\nabla_0 A(z) - (2i\mu)^2 R_0(z)R_0(\bar{z})A(\bar{z})R_0(z)R_0(\bar{z})\nabla_0 A(z), \quad (11)$$

where the operator $\nabla_0 A(z)$ is given by the kernel

$$\nabla_0 A(z) \sim \rho_0^{-1}(P'^2) \left(\frac{\partial}{\partial \lambda} + \frac{\partial}{\partial \bar{P}^2} + \frac{\partial}{\partial P'^2} \right) \rho_0(P'^2) A(P, P'; \lambda + i\mu), \quad (12)$$

where $A(P, P'; z)$ is the kernel of the operator $A(z)$ in $L_2(R^6)$; differentiation with respect to P^2 is carried out for fixed $(P^2)^{-1/2}P = \omega_0$; the operations with respect to P' are defined analogously.

Preparatory formula for the junior channels:

$$\text{Sp } \mathfrak{R}_1(z) = \text{Sp} \left\{ Q[T(z)] - \sum_{\alpha} Q[T_{\alpha}(z)] \right\}, \quad (13)$$

where

$$\begin{aligned} Q[T(z)] = & -(2i\mu)^2 \sum_{\alpha, \beta, \beta', \alpha'} R_0(z) R_0(\bar{z}) M_{\alpha\beta}(z) R_0(z) R_0(\bar{z}) \nabla_{\alpha\beta\beta'\alpha'} M_{\beta'\alpha'}(z) + \\ & + 2i\mu \left\{ \sum_{\alpha \neq \alpha'} [M_{\alpha\alpha'}(z) - M_{\alpha\alpha'}(\bar{z})] \left[\frac{d}{d\lambda} R_0(z) R_0(\bar{z}) \right] - \frac{d}{d\lambda} [R_0(z) R_0(\bar{z}) T(z)] \right\}. \end{aligned} \quad (14)$$

Here $M_{\alpha\beta}(z)$ ($\alpha, \beta = 1, 2, 3$) are the operators introduced in (1):

$$M_{\alpha\beta}(z) = \delta_{\alpha\beta} V_{\alpha} - V_{\alpha} R(z) V_{\beta}, \quad V_{\alpha} = H_{\alpha} - H_0,$$

where

$$T(z) = \sum_{\alpha, \beta} M_{\alpha\beta}(z).$$

Further: the operator $\nabla_{\alpha\beta\beta'\alpha'} M_{\beta'\alpha'}(z)$ is given by the kernel

$$\begin{aligned} & \left\{ \frac{\partial}{\partial \lambda} + \delta_{\beta\beta'} 2n_{\beta} \frac{\partial}{\partial p_{\beta}^2} + \right. \\ & \left. + \delta_{\alpha\alpha'} \rho_{\alpha}^{-1} \left(\frac{p_{\alpha}^2}{2n_{\alpha}} \right) \cdot 2n_{\alpha} \frac{\partial}{\partial p_{\alpha}^2} \rho_{\alpha} \left(\frac{p_{\alpha}^2}{2n_{\alpha}} \right) \right\} M_{\beta'\alpha'}(P, P'; \lambda + i\mu), \end{aligned} \quad (15)$$

where M with arguments is the kernel of $M(z)$. Differentiation with respect to p_α^2 is carried out for fixed

$$\omega_\alpha = (2n_\alpha)^{1/2} \frac{p_\alpha}{|p_\alpha|}$$

and k_α . The pairs of variables $\{k_\alpha, p_\alpha\}$ used to describe P are introduced in the same way as in (1).

4. Trace formula. The result of passing to the limit $\mu \rightarrow 0$ in formulas (10) and (13) is determined by the singularities of the kernel of the operator $T(z)$ (see (1)).

Trace formula for the junior channels. Let $\lambda < 0$, $\lambda \in \Phi$; then, as $\mu \rightarrow 0$, in (13) only the first terms in $Q[T(z)]$, corresponding to $\alpha = \alpha'$, $\beta = \beta'$, are retained.

Theorem 1. *In the limit we obtain*

$$\Omega_\pm(\lambda) = \pm \text{Sp } \mathfrak{S}_{(\lambda)} S^*(\lambda) dS(\lambda)/d\lambda. \quad (16)$$

Here $dS(\lambda)/d\lambda$ is given by differentiated kernels, as described in Sec. 2. The trace on the right-hand side of (16) is understood as the trace of the kernel.

Trace formula for the senior channel. Let $\lambda > 0$, $\lambda \in \Sigma_0$. For simplicity we shall assume the scattering to be single-channel, i.e.

$$\mathfrak{S}_1 \cup \mathfrak{S}_2 \cup \mathfrak{S}_3 = \emptyset.$$

Put

$$\dot{T}(z) = \sum_{\alpha} T_{\alpha}(z) - \sum_{\alpha \neq \beta} T_{\alpha}(z) R_0(z) T_{\beta}(z).$$

$\dot{T}(z)$ contains the senior singularities of the kernel $T(z)$. The operator $S(\lambda)$ is expressed explicitly in terms of $T(z)$. We denote by $\dot{S}(\lambda)$ the contribution to $S(\lambda)$ from $\dot{T}(z)$.

Theorem 2. Passing to the limit in formula (10) gives

$$\Omega_\pm(\lambda) = \pm \text{Sp}_{\mathcal{C}(\lambda)} \left[S^*(\lambda) \frac{dS(\lambda)}{d\lambda} - \dot{S}^*(\lambda) \frac{d\dot{S}(\lambda)}{d\lambda} \right] \pm \Delta(\lambda), \quad (17)$$

where

$$\Delta(\lambda) = \lim_{\mu \downarrow 0} \text{Sp} \left\{ \mathfrak{P}[\dot{T}(z)] - \sum_a \mathfrak{P}[T_a(z)] \right\}. \quad (18)$$

Here the same explanations must be made as in Theorem 1. It is clear that $\Delta(\lambda)$ is a polynomial in the “pair” operators $T_a(z)$, which enter into $\Delta(\lambda)$ at most to the fourth degree. They are explicitly expressed in terms of the resolvents of the operators h_a .

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