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Abstract

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MATHEMATICS

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EXPONENTIAL STABILIZATION OF LINEAR SYSTEMS

(Presented by Academician V. I. Smirnov on 4 X 1968)

Methods are well known for determining the parameters entering into the coefficients of a linear system of differential equations for which the solutions of the system satisfy the estimate $|\mathbf{x}(t)| \leq Ce^{-\alpha t}|\mathbf{x}(0)|$ with a prescribed $\alpha \geq 0$. In this, however, the constant C remains, generally speaking, undetermined, which reduces the practical significance of these methods. In the present paper we consider the problem (under a number of assumptions) of choosing the parameters of the system for which the estimate $|\mathbf{x}(t)| < e^{-\alpha t}|\mathbf{x}(0)|$ is valid, where $\alpha \geq 0$ and the norm $|\mathbf{x}| = \sqrt{\mathbf{x}^*H\mathbf{x}}$ ($H = H^* > 0$) are regarded as prescribed*.

1°. Consider the linear control system

$$dx/dt = Ax + Ru, \quad \mathbf{u} = -S^*\mathbf{x} \quad (1)$$

and the positive definite form

$$V(\mathbf{x}) = \mathbf{x}^*H\mathbf{x}, \quad H = H^* > 0. \quad (2)$$

Here $\mathbf{x}$ is an n -dimensional state vector; $\mathbf{u}$ is a control vector of order m , $m \leq n$ (the number m is called below the number of controls); A, H, R, S are constant real matrices of orders $n \times n$, $n \times n$, $n \times m$, $n \times m$, respectively; the asterisk denotes transposition.

The problem is to determine, from given matrices A, H, R (or A, H, S) and a number $\alpha \geq 0$, a matrix S (or R) such that, on every solution of system (1), the inequality

$$\dot{V}(\mathbf{x}) + 2\alpha V(\mathbf{x}) < 0, \quad \mathbf{x} \neq 0 \quad (3)$$

is satisfied.

If inequality (3) is satisfied, then we shall say that the matrices R, S α -stabilize the system $dx/dt = Ax$ with respect to the form (2), or, more briefly, that

system (1), (2) is α -stabilized. In this case we shall call the matrix S (or R) paired with the matrix R (or S), and the matrix R (or S) for which a paired one exists will be called admissible.

We note that if the control system is α -stabilized, then it is asymptotically stable. The converse (for the given form (2)), obviously, is not true**.

Consider the matrix C_α of the quadratic form $-\dot{V}(x) - 2\alpha V(x)$, where $\dot{V}(\mathbf{x})$ is computed by virtue of the first equation (1) in the absence of control ($u = 0$): $C_\alpha = -(A^*H + HA + 2\alpha H)$. Suppose that the form $\mathbf{x}^*C_\alpha\mathbf{x}$ is indefinite and has $\mu_\alpha \geq 1$ nonpositive squares***.

Theorem 1. *In order that the system $dx/dt = Ax$ be capable of being α -stabilized with respect to the form (2), it is necessary that the number of controls m and the number μ_α be related by the inequality $m \geq \mu_\alpha$.*

The theorems 2-4 formulated below give conditions for admissibility of the matrix R and determine the form of the paired matrix S .

Let L be an arbitrary $n \times k$ matrix of rank k . By $\{L\}_k$ we shall denote the subspace spanned by its columns. For a given subspace—

* The notation $K = K^* > 0$ means that the form $\mathbf{x}^*K\mathbf{x}$ is positive definite.

** Recall that $\alpha \geq 0$. What follows is also valid for $\alpha < 0$.

*** For $\mu_\alpha = 0$ the problem admits the trivial solution $S = 0$ (or $R = 0$).

the space $\{L\}_k$ of dimension k will be denoted by L , an arbitrary matrix whose columns form a basis in $\{L\}_k$. Consider the subspace $\{HR\}_m$, its orthogonal complement $\{Z\}_{n-m}$, and the corresponding matrix Z . Obviously, $Z^*HR = 0$.

Theorem 2. Let R be a given matrix of order $n \times m$ and rank m . In order that the matrix R be admissible, it is necessary and sufficient that the matrix $Z^*C_\alpha Z$ be positive definite.

Suppose that the matrix $Z^*C_\alpha Z$ is not positive definite and has ν ($0 < \nu < n - m$) nonpositive eigenvalues. By Theorem 2, the matrix R is not admissible. Theorem 3 answers the question of how one can “complete” the matrix R by adjoining to it a matrix R_2 with the minimal number of columns (i.e., by minimally increasing the number of controls), so that the resulting matrix $R_1 = \|R, R_2\|$ becomes admissible. Consider a subspace $\{Z_0\}_{n-m-\nu} \subset \{Z\}_{n-m}$ such that $Z_0^*C_\alpha Z_0 > 0$, and construct its orthogonal complement $\{Y\}_\nu \subset \{Z\}_{n-m}$, $Y^*Z_0 = 0$.

Theorem 3. Let the matrix $Z^*C_\alpha Z$ have ν ($0 < \nu < n - m$) nonpositive eigenvalues. The system (1), (2) can be α -stabilized by introducing, in addition, ν controls. In this case the matrix $R_1 = \|R, H^{-1}Y\|$ is admissible.

Theorem 4. Let the matrix R of order $n \times m$ be admissible. The general form of a matrix S paired with it is given by the formula $S = \lambda HR + U$, where $\lambda > \lambda_0 > 0$ is an arbitrary number for which the inequality $C_\alpha + \lambda HRR^*H > 0$ is satisfied, and U is an arbitrary matrix of order $n \times m$ satisfying the condition

$UU^* < \lambda(C_\alpha + \lambda HRR^*H)$. Here and below, λ_0 is the minimal number such that $C_\alpha + \lambda_0 HRR^*H \geq 0$.

Since inequality (3) is symmetric with respect to the matrices HR and S , there are theorems, analogous to Theorems 2-4, that solve the problem of choosing the matrix S .

Consider a problem close to the one solved above. Let, in the system $dx/dt = Ax + Ru$, $\mathbf{v} = -S^*\mathbf{x}$, the matrices A, R, S be given and let the closed-loop system for $\mathbf{u} = \mathbf{v}$ not be α -stabilized. Consider the question of whether the system can be α -stabilized by introducing the connection $\mathbf{u} = \tau^*\mathbf{v}$, where τ is some constant $m \times m$ matrix.

Theorem 5. Let the matrices A, R, S be given and let R be an admissible matrix. Put $K = \lambda(C_\alpha + \lambda HRR^*R)$, $\sigma = \lambda(S^*K^{-1}S)^{-1}SK^{-1}HR$, $U_1 = S\sigma - \lambda HR$, where $\lambda > \lambda_0 > 0$ is an arbitrary number for which the inequality $K > 0$ is satisfied. In order that the matrix R have a paired matrix S_1 of the form $S_1 = S\tau$, where τ is some $m \times m$ matrix, it is necessary and sufficient that the inequality $U_1U_1^* < K$ be satisfied.

When the last inequality is satisfied, one can take $S_1 = S\sigma$.

2°. In the following lemmas, the vectors x and y have orders n and m , respectively; F is a matrix of order $n \times m$ and rank m .

Lemma 1. In order that the quadratic form $O(x, y)$ be positive definite on the subspace $F^*x = y$, it is necessary and sufficient that there exist $\lambda_0 > 0$ such that the form $Q(x, y) + \lambda(x^*F - y^*)(F^*x - y)$ is positive definite on the whole space $\{x, y\}$ for $\lambda > \lambda_0$.

The proof of this, apparently known, lemma, obtained by the author jointly with I. E. Zuber, is omitted here.

Lemma 2. Let x^*Cx be an indefinite form. For the existence of a number $\lambda_0 > 0$ such that the inequality $M = C + \lambda FF^* > 0$ is satisfied for all $\lambda > \lambda_0$, it is necessary and sufficient that, for an arbitrary matrix Y of order $n \times (n - m)$ and rank $n - m$ such that $Y^*F = 0$, the inequality $Y^*CY > 0$ hold.

Necessity is obvious.

Sufficiency. Let $Y^*CY > 0$. Consider the matrix $T = \|T_F, T_Y\|$, the first m columns of which form an orthonormal basis in the sub-

space spanned by the column vectors of the matrix F , and the last $n - m$ columns form an orthonormal basis in the subspace that is the orthogonal complement of the first. Then, obviously, $T^* = T^{-1}$ and

$$F_0 = T^*F = \begin{Bmatrix} \Phi \\ 0 \end{Bmatrix}, \quad Y_0 = T^*Y = \begin{Bmatrix} 0 \\ \Psi \end{Bmatrix}.$$

Here Φ, Ψ are nonsingular matrices of orders $m \times m$ and $m \times (n - m)$.

Let us partition the matrix $C_0 = T^*CT$ into blocks $C_{11}, C_{12} = C_{21}^*, C_{22}$ of orders, respectively, $m \times m, m \times (n - m), (n - m) \times (n - m)$. It is clear that $Y^*CY = Y_0^*C_0Y_0$ and that the inequality $Y^*CY > 0$ has the form $\Psi^*C_{22}\Psi > 0$. Hence $C_{22} > 0$. Write the matrix $M_0 = T^*MT$ in block form:

$$M_0 = \begin{bmatrix} C_{11} + \lambda\Phi\Phi^* & C_{12} \\ C_{12}^* & C_{22} \end{bmatrix}.$$

Obviously, the condition $C_{22} > 0$ is sufficient (and necessary) for there to exist $\lambda_0 > 0$ such that for $\lambda > \lambda_0$ the inequality $M_0 > 0$, and hence also the inequality $M > 0$, is satisfied.

Proof of Theorem 1. Put $R^*Hx = y$. From $V(x) + \alpha\dot{V}(x) = -Q(x, y)$ we obtain $Q(x, y) = x^*C_\alpha x + y^*S^*x + x^*Sy$. Inequality (3) is equivalent to positive definiteness of the form $Q(x, y)$ on the subspace $R^*Hx = y$. By Lemma 1, for inequality (3) to hold it is necessary and sufficient that there exist $\lambda_0 > 0$ such that for $\lambda > \lambda_0$ the form

$$Q(x, y) + \lambda(x^*HR - y^*)(R^*Hx - y)$$

is positive definite for all $\{x, y\}$. It is easily shown that the latter condition is equivalent to the fact that, for $\lambda > \lambda_0$,

$$C_\alpha + \lambda HRR^*H > 0.$$

Hence, by virtue of ([1], Ch. X, Theorem 16 ($A(x, x) = x^*C_\alpha x$, $\tilde{A}(x, x) = A(x, x) + \lambda x^*HRR^*Hx$, $B(x, x) = x^*x$, $m = r$)), the conclusion of the theorem follows.

Proof of Theorem 2. As shown above, the condition

$$C_\alpha + \lambda HRR^*H > 0$$

is necessary and sufficient for the admissibility of the matrix R . By Lemma 2 this inequality is equivalent to the inequality $Z^*C_\alpha Z > 0$, which proves the theorem.

Proof of Theorem 4. Since R is an admissible matrix, there exists $\lambda_0 > 0$ such that for arbitrary $\lambda > \lambda_0$ and some matrix S^* the inequality

$$Q(x, y) + \lambda(x^*HR - y^*)(R^*Hx - y) > 0$$

is satisfied, or, equivalently to it, the inequalities

$$C_\alpha + \lambda H R R^* H > 0, \quad \lambda(C_\alpha + \lambda H R R^* H) - (S - \lambda H R)(S^* - \lambda R^* H) > 0.$$

Putting $U = S - \lambda H R$, we obtain the assertion of the theorem.

Proof of Theorem 3. By assumption the matrix $Z^* C_\alpha Z$ has exactly $n - m - \nu$ positive eigenvalues. Therefore, by Theorem 2, the number of controls by which one can α -stabilize the system (1), (2) is no less than $m + \nu$. It is easy to see that it can be equal to $m + \nu$. Indeed, the matrix $R_1 = \|R, H^{-1}Y\|$ has rank $m + \nu$ and is admissible: $Z_0^* H R_1 = \|Z_0^* H R, Z_0^* Y\|$ and $Z_0^* C_\alpha Z_0 > 0$.

Proof of Theorem 5. Necessity. Let the matrices R and S α -stabilize the system (1), (2); let $\lambda > \lambda_0$ be such that $K > 0$ and $U = S\tau - \lambda H R$. By Theorem 4, $U U^* < K$. Since $K > 0$, the inequality $U U^* < K$ is equivalent to the inequality

$$U^* K^{-1} U < I_m.$$

Put $\tau = \sigma + \tau_0$; then $U = U_1 + S\tau_0$. Here the matrices U_1, σ, K have the form indicated in Theorem 5. Since $S^* K^{-1} U_1 = 0$, it follows that

$$U_1^* K^{-1} U_1 = U^* K^{-1} U - \tau_0^* S^* K^{-1} S \tau_0 < I_m.$$

Therefore $U_1 U_1^* < K$.

Sufficiency. From Theorem 4, for $U = U_1$ we obtain that $S_1 = S\sigma$ is a pairing matrix.

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REFERENCES

1. F. R. Gantmakher, *Matrix Theory*, Nauka, 1967.

* The matrix S may be both given and sought.

Note: Figure translations are in progress. See original paper for figures.

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