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Abstract

Full Text

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MATHEMATICS

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ON A GENERALIZATION OF THE INTEGRAL FORMULAS OF CAUCHY, SCHWARZ, AND POISSON

(Presented by Academician M. A. Lavrent'ev on 5 February 1969)

M. M. Dzhrbashyan ⁽¹⁾ constructed a generalized operator $L^{(\omega)}$ of Riemann-Liouville type, by means of which he established fundamentally new analogues of the classical formulas of Cauchy, Schwarz, and Poisson*—generalized Cauchy, Schwarz, and Poisson formulas associated with a given function $\omega(x) \in \Omega^{**}$. In the present note this generalized operator is used to establish (Theorems 1, 2) generalized Cauchy, Schwarz, and Poisson formulas associated with a given system of functions $\omega_j(x) \in \Omega$ ($j = 1, 2, \dots, m$).

Let the functions $\omega_j(x) \in \Omega$ ($j = 1, 2, \dots, m$). Further, let $p_j(0) = 1$,

$$p_j(r) = r \int_r^1 \frac{\omega_j(x)}{x^2} dx \quad (r \in (0, 1]), \quad \Delta_0^{(j)} = 1, \quad \Delta_k^{(j)} = -(k+1) \int_0^1 r^k dp_j(r) =$$

$$= k \int_0^1 r^{k-1} \omega_j(r) dr \quad (j = 1, 2, \dots, m), \quad k = 1, 2, \dots^{***}.$$

We introduce for consideration the power series

$$C(z; \omega_1, \dots, \omega_m) = \sum_{k=0}^{\infty} \frac{z^k}{\Delta_k^{(1)} \dots \Delta_k^{(m)}}. \quad (1)$$

It is easy to see that the radius of convergence of this series is equal to one. Thus the function $C(z; \omega_1, \dots, \omega_m)$ is holomorphic in the disk $|z| < 1$. Along with this function we also introduce the function

$$S(z; \omega_1, \dots, \omega_m) = 2C(z; \omega_1, \dots, \omega_m) - C(0; \omega_1, \dots, \omega_m) =$$

$$= 1 + 2 \sum_{k=1}^{\infty} \frac{z^k}{\Delta_k^{(1)} \dots \Delta_k^{(m)}}, \quad (2)$$

noting that $C(0; \omega_1, \dots, \omega_m) = 1/\Delta_0^{(1)} \dots \Delta_0^{(m)} = 1$.

* For other generalizations of the Cauchy, Schwarz, and Poisson formulas, see, for example, ⁽²⁻⁶⁾.

** It is said (see ⁽¹⁾, p. 1078) that a function $\omega(x) \in \Omega$ if it is nonnegative and continuous on $[0, 1]$, with $\omega(0) = 1$,

$$\int_0^1 \omega(x) dx < +\infty$$

and for every r ($0 \leq r < 1$)

$$\int_r^1 \omega(x) dx > 0.$$

*** In ⁽¹⁾ the function $p(0) = 1$,

$$p(r) = r \int_r^1 \frac{\omega(x)}{x^2} dx \quad (\omega(x) \in \Omega), \quad r \in (0, 1],$$

and the sequence of numbers

$$\Delta_k = -(k+1) \int_0^1 r^k dp(r) \quad (k = 0, 1, 2, \dots)$$

were introduced, and it was shown that all the numbers Δ_k ($k = 0, 1, 2, \dots$) are positive, with $\Delta_0 = 1$,

$$\Delta_k = k \int_0^1 \omega(x) x^{k-1} dx \quad (k = 1, 2, \dots).$$

Theorem 1. Let the function

$$f(re^{i\varphi}) = \sum_{k=0}^{\infty} a_k (re^{i\varphi})^k$$

be holomorphic in the disk $|z| < R$. Then the function

$$\begin{aligned}
 & L^{(\omega_m)} [L^{(\omega_{m-1})} \dots [L^{(\omega_1)} [f(re^{i\varphi})]] \dots] \equiv \\
 & \equiv L^{(\omega_1, \dots, \omega_m)} [f(re^{i\varphi})] \equiv f_{(\omega_1, \dots, \omega_m)}(re^{i\varphi}) = \sum_{k=0}^{\infty} \Delta_k^{(1)} \dots \Delta_k^{(m)} a_k (re^{i\varphi})^k \quad (3)
 \end{aligned}$$

is holomorphic in the same disk $|z| < R$, and for any ρ ($0 < \rho < R$) the integral formulas

$$\begin{aligned}
 f(z) &= \frac{1}{2\pi} \int_0^{2\pi} C\left(e^{-i\theta} \frac{z}{\rho}; \omega_1, \dots, \omega_m\right) f_{(\omega_1, \dots, \omega_m)}(\rho e^{i\theta}) d\theta \quad (|z| < \rho), \\
 f(z) &= i \operatorname{Im} f(0) + \frac{1}{2\pi} \int_0^{2\pi} S\left(e^{-i\theta} \frac{z}{\rho}; \omega_1, \dots, \omega_m\right) \times \\
 &\quad \times \operatorname{Re} f_{(\omega_1, \dots, \omega_m)}(\rho e^{i\theta}) d\theta \quad (|z| < \rho).
 \end{aligned}$$

In the course of the proof, the Cauchy-Hadamard formula is essentially used, as well as the formulas

$$L^{(\omega_1, \dots, \omega_m)}[r^k] = \Delta_k^{(1)} \dots \Delta_k^{(m)} r^k \quad (k = 0, 1, 2, \dots)$$

and the expansions (1), (2), and (3).

Let us introduce into consideration the function

$$P(\theta, r; \omega_1, \dots, \omega_m) = \operatorname{Re} S(re^{i\theta}; \omega_1, \dots, \omega_m) = 1 + 2 \sum_{k=1}^{\infty} \frac{r^k \cos k\theta}{\Delta_k^{(1)} \dots \Delta_k^{(m)}},$$

harmonic in the unit disk $0 \leq r < 1$, $0 \leq \theta \leq 2\pi$.

From Theorem 1 it follows easily:

Theorem 2. Let $u(z)$ be a harmonic function in the disk $|z| < R$. Then the function

$$u_{(\omega_1, \dots, \omega_m)}(re^{i\varphi}) = L^{(\omega_1, \dots, \omega_m)}[u(re^{i\varphi})]$$

will be harmonic in the same disk $|z| < R$, and for any ρ ($0 < \rho < R$) the integral formula

$$u(re^{i\varphi}) = \frac{1}{2\pi} \int_0^{2\pi} P\left(\varphi - \theta, \frac{r}{\rho}; \omega_1, \dots, \omega_m\right) u_{(\omega_1, \dots, \omega_m)}(\rho e^{i\theta}) d\theta$$

holds,

$$(0 \leq r < \rho, \quad 0 \leq \varphi \leq 2\pi).$$

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Note: Figure translations are in progress. See original paper for figures.

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