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1969

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Abstract

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UDC 519.44

MATHEMATICS

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ON COMPOSITION BLOCKS OF FINITE GROUPS

§ 1. In papers ^(1,2) (see also ⁽³⁾) we indicated a method of representing the order $|\mathfrak{G}|$ of an arbitrary finite group $\mathfrak{G}$ in the form of such a product

$$|\mathfrak{G}| = m_1 m_2 \cdots m_k, \quad k \geq 1,$$

of pairwise relatively prime factors $m_1, m_2, \dots, m_k$, to which there corresponds a factorization

$$\mathfrak{G} = \mathfrak{M}_1 \mathfrak{M}_2 \cdots \mathfrak{M}_k$$

of the group $\mathfrak{G}$ by means of certain pairwise permutable subgroups $\mathfrak{M}_1, \mathfrak{M}_2, \dots, \mathfrak{M}_k$ of orders $m_1, m_2, \dots, m_k$, respectively. Moreover, for solvable $\mathfrak{G}$ the system of subgroups $\mathfrak{M}_1, \mathfrak{M}_2, \dots, \mathfrak{M}_k$ automatically becomes a Sylow system of P. Hall.

We called ^(1,2) the systems of subgroups $\mathfrak{M}_1, \mathfrak{M}_2, \dots, \mathfrak{M}_k$ complete systems of permutable components of the group $\mathfrak{G}$, the numbers $m_1, m_2, \dots, m_k$ composition blocks of $\mathfrak{G}$ (their definition is given in § 3), and the factorization found by us a composition-block factorization of finite groups ⁽³⁾.

In the present paper it is shown (in Theorem 1) that it is possible to factor a finite group by any set of subgroups on whose orders of intersection with the members of a given invariant series of the group a certain "embedding condition" is imposed, and a number of consequences are derived from this. In particular, we prove that an arbitrary finite group is factored not only by means of complete systems of pairwise permutable components, but also by means of any set of subgroups of orders $m_1, m_2, \dots, m_k$. Let us note that in the results given below, factorizations are established not with pairwise permutability of the factors—which is impossible in the general case of the sets of subgroups considered by us—but with permutability of them as a whole. Therefore such factorizations may be called totally permutable. On the other hand, the results obtained by us may also be regarded as criteria for the total permutability of certain subgroups (cf. ⁽⁸⁾).

In § 4 the natural place in the theory of finite groups of the class of π -solvable groups introduced by us in ⁽⁴⁻⁶⁾ is revealed.

§ 2. We give a list of the notation and definitions needed by us. $\mathfrak{G}$ is an arbitrary finite group; $|\mathfrak{G}|$ is its order; p is a prime number; π is a set of prime numbers (empty or not); a p -solvable group ⁽⁴⁾ is a finite group in which each index divisible by p of a composition series is equal to p ; a π -solvable group ^(5,6) is a group that is p -solvable for every $p \in \pi$; $\mathfrak{E}$ is the identity subgroup of $\mathfrak{G}$; we classify $\mathfrak{G}$ as solvable, supersolvable, and nilpotent groups, and as Sylow subgroups of $\mathfrak{G}$; the series $\mathfrak{C}, \mathfrak{C}'$ are regarded as composition, chief, and characteristic for $\mathfrak{G}$; an invariant series of a group is a series all of whose members are invariant subgroups of the group; N is a sequence of natural numbers, among which there may also be equal ones; $\overline{N}$, for N nonempty, we set equal to the product of all elements of N , and for N empty, to 1; a primary number is a power of a prime number (including the zeroth power); if n is a natural number, then $\pi(n)$ is the set of all distinct prime divisors of n ; a π -number is a natural number for which $\pi(n) \subseteq \pi$; a π -group (subgroup) is a group (subgroup) whose order is a π -number; a π -maximal subgroup is a π -subgroup that is not a proper subgroup of any π -subgroup; a π -divisor is a divisor that is a π -number.

§ 3. Let $n_1 \in N$ and let $N_1^1 = \{n_1\}$. Construct the sequence of sets $N_1^1, N_1^2, \dots$, in which each N_1^{i+1} , $i \geq 1$, is the union of N_1^i and the set of all those elements of N that are not relatively prime to at least one of the elements of N_1^i . Obviously, this sequence stabilizes at some term $N_1^{r_1} = N_1$. If N_1 does not yet exhaust N , then take $n_2 \in N \setminus N_1$ and repeat the procedure indicated above. As a result, N decomposes into pairwise disjoint classes $N_1, N_2, \dots, N_k$, $k \geq 1$, of arithmetically connected elements ⁽¹⁾. The numbers $m_1 = \overline{N_1}, m_2 = \overline{N_2}, \dots, m_k = \overline{N_k}$ will be called the blocks of the sequence N . In ⁽¹⁾ we called the blocks of the sequence of indices of any composition series of an arbitrary finite group $\mathfrak{G}$ the **composition blocks** of the group $\mathfrak{G}$.

A divisor h of the number $|\mathfrak{G}|$ will be called full-block ⁽²⁾ if $h = \overline{M}$, where M is some (empty or not) set of composition blocks of the group $\mathfrak{G}$.

§ 4. From the assertion on the solvability of groups of odd order it follows that the composition-block factorization $\mathfrak{G} = \mathfrak{M}_1 \mathfrak{M}_2 \dots \mathfrak{M}_k$ contains no more than one factor of even order, while the remaining factors (if they exist) are subgroups of primary odd orders. Therefore $\mathfrak{G} = \mathfrak{G} \mathfrak{G}$, with $|\mathfrak{G}| = \overline{M}$ and $|\mathfrak{G}| = \overline{M}$, where M and $\overline{M}$ are the sets of all even and all primary composition blocks of $\mathfrak{G}$. The subgroup $\mathfrak{G}$ (the even component of $\mathfrak{G}$) is single-block (Lemma 7 from ⁽²⁾); the subgroup $\mathfrak{G}$ (the primary component of $\mathfrak{G}$) is solvable. It is also obvious that $\mathfrak{G}$ is $\pi(\overline{M})$ -solvable for any $M \subseteq \overline{M}$. Conversely, if $\mathfrak{G}$ is π -solvable, then the greatest π -divisor of $|\mathfrak{G}|$ is equal to $\overline{M}$, where $M \subseteq \overline{M}$. If h is any full-block divisor of $|\mathfrak{G}|$, then for $\mathfrak{G}$ the $\pi(h)$ -Sylow theorem holds, i.e. all $\pi(h)$ -maximal subgroups of $\mathfrak{G}$ are conjugate ⁽⁷⁾.

§ 5. **Definition 1.** Subgroups $\mathfrak{A}_1, \mathfrak{A}_2, \dots, \mathfrak{A}_\mu$, $\mu \geq 1$, of $\mathfrak{G}$ will be called **totally permutable** if the product $\mathfrak{A}_1 \mathfrak{A}_2 \dots \mathfrak{A}_\mu$ does not depend on the order of the factors (then it will also be a subgroup of $\mathfrak{G}$).

Let now

$$\mathfrak{G} = \mathfrak{G}_0 \supseteq \mathfrak{G}_1 \supseteq \dots \supseteq \mathfrak{G}_\lambda = \mathfrak{E}$$

be some invariant series of the group $\mathfrak{G}$ with the sequence of indices $N_{\mathfrak{G}} = \{h_1, h_2, \dots, h_\lambda\}$, which, for $\mathfrak{G} \neq \mathfrak{E}$, has no repetitions. (1)

Let also H_j be the set of all elements of $N_{\mathfrak{G}}$ whose numbers are greater than j .

Let

$$N_{\mathfrak{G}} = \bigcup_{i=1}^{\mu} M_i, \quad \mu \geq 1,$$

be some arbitrary representation of the set $N_{\mathfrak{G}}$ as the union of some of its subsets M_i , and let to each M_i , $i = 1, 2, \dots, \mu$, there be assigned some subgroup $\mathfrak{A}_i$ of $\mathfrak{G}$ satisfying the following “embedding condition” :

$$|\mathfrak{A}_i \cap \mathfrak{G}_j| = \overline{D}_{ij} d_{ij}, \quad \text{where } D_{ij} = M_i \cap H_j, \quad i = 1, 2, \dots, \mu; \quad j = 0, 1, \dots, \lambda,$$

$$\text{and, if } j < \lambda, \text{ then } d_{ij} \text{ is divisible by } d_{i,j+1}. \quad (2)$$

Definition 2. Any collection of subgroups satisfying condition (2) will be called a **properly embedded system of subgroups** of the group $\mathfrak{G}$.

Definition 3. If for a collection of subgroups $\mathfrak{B}_1, \mathfrak{B}_2, \dots, \mathfrak{B}_\nu$, $\nu \geq 1$, of the group $\mathfrak{G}$ the equality $\mathfrak{G} = \mathfrak{B}_1 \mathfrak{B}_2 \dots \mathfrak{B}_\nu$ holds, then we shall say that it σ -factorizes the group, where σ is the above-indicated arrangement of the factors (when the arrangement of the factors is immaterial, the sign σ will be omitted).

It is obvious that if a given system of subgroups of the group $\mathfrak{G}$ factorizes $\mathfrak{G}$ (and is totally permutable), then this is also true for every system of subgroups from $\mathfrak{G}$ containing it.

Theorem 1. *Every properly embedded system of subgroups of an arbitrary finite group $\mathfrak{G}$ is totally permutable and factorizes $\mathfrak{G}$.*

Proof. Suppose that the theorem is false, and let the group $\mathfrak{G}$ then be a counterexample of least order. Clearly, then $\lambda > 1$, $\mu > 1$, and hence also $|\mathfrak{G}| > 1$. Then $\mathfrak{G}_{\lambda-1} \neq \mathfrak{G}$ and $|\mathfrak{G}/\mathfrak{G}_{\lambda-1}| < |\mathfrak{G}|$. Consider now the group $\mathfrak{A}_i \mathfrak{G}_{\lambda-1} / \mathfrak{G}_{\lambda-1}$, $i = 1, 2, \dots, \mu$. Its order is obviously equal to $a'_i = |\mathfrak{A}_i| / \overline{D}_{i\lambda-1} d_{i\lambda-1}$.

If $h_\lambda \in M_i$, then

$$a'_i = (\overline{M_i \setminus \{h_\lambda\}}) d_{i0} / d_{i\lambda-1},$$

whereas if $h_\lambda \notin M_i$, then

$$a'_i = \overline{M_i} d_{i0} / d_{i\lambda-1}.$$

In the first case put $M'_i = M_i \setminus \{h_\lambda\}$, and in the second $M'_i = M_i$. Then

$$\bigcup_{i=1}^{\mu} M'_i = N_{\mathfrak{G}} \setminus \{h_\lambda\} = N_{\mathfrak{G}/\mathfrak{G}_{\lambda-1}},$$

where $N_{\mathfrak{G}/\mathfrak{G}_{\lambda-1}}$ is the sequence of indices of the invariant series

$$\mathfrak{G}/\mathfrak{G}_{\lambda-1} = \mathfrak{G}_0/\mathfrak{G}_{\lambda-1} \supset \mathfrak{G}_1/\mathfrak{G}_{\lambda-1} \supset \dots \supset \mathfrak{G}_{\lambda-1}/\mathfrak{G}_{\lambda-1}$$

of the group $\mathfrak{G}/\mathfrak{G}_{\lambda-1}$. Moreover, for $i = 1, 2, \dots, \mu$ and $j = 0, 1, \dots, \lambda - 1$ we have:

$$\eta_{ij} = |\mathfrak{A}_i \mathfrak{G}_{\lambda-1}/\mathfrak{G}_{\lambda-1} \cap \mathfrak{G}_j/\mathfrak{G}_{\lambda-1}| = |(\mathfrak{A}_i \cap \mathfrak{G}_j) \mathfrak{G}_{\lambda-1}/\mathfrak{G}_{\lambda-1}| = \bar{D}_{ij} d_{ij} / |\mathfrak{A}_i \cap \mathfrak{G}_{\lambda-1}|.$$

Consider again the two possible cases.

- 1) $h_\lambda \in M_i$. Then it is clear that $D_{i\lambda-1} = \{h_\lambda\}$ and that $h_\lambda \in D_{ij}$. In this case

$$|\mathfrak{A}_i \cap \mathfrak{G}_{\lambda-1}| = h_\lambda d_{i\lambda-1} = h_\lambda;$$

$$\eta_{ij} = (\bar{D}_{ij} \setminus \{h_\lambda\}) d_{ij} / d_{i\lambda-1}$$

and

$$D_{ij} \setminus \{h_\lambda\} = M'_i \cap H_j.$$

- 2) $h_\lambda \notin M_i$. Then

$$\eta_{ij} = \bar{D}_{ij} d_{ij} / d_{i\lambda-1}$$

and

$$D_{ij} = M'_i \cap H_j.$$

The divisibility condition from (2) shows that for $j < \lambda - 1$ the number $d_{ij}/d_{i\lambda-1}$ is divisible by $d_{ij+1}/d_{i\lambda-1}$. We see that, for the system of subgroups

$$\mathfrak{A}_i \mathfrak{G}_{\lambda-1}/\mathfrak{G}_{\lambda-1}, \quad i = 1, 2, \dots, \mu,$$

of the group $\mathfrak{G}/\mathfrak{G}_{\lambda-1}$, all conditions of the theorem being proved are satisfied.

Since $|\mathfrak{G}/\mathfrak{G}_{\lambda-1}| < |\mathfrak{G}|$, the theorem is true for $\mathfrak{G}/\mathfrak{G}_{\lambda-1}$, i.e., the system of subgroups

$$\mathfrak{A}_i \mathfrak{G}_{\lambda-1}/\mathfrak{G}_{\lambda-1}, \quad i = 1, 2, \dots, \mu,$$

is totally permutable and factorizes $\mathfrak{G}/\mathfrak{G}_{\lambda-1}$. But then there is a factorization

$$\mathfrak{G} = \mathfrak{A}_1 \mathfrak{A}_2 \dots \mathfrak{A}_\mu \mathfrak{G}_{\lambda-1}$$

with totally permutable factors. By the hypothesis, there exists at least one M_ν containing h_λ . As we saw above, then $\mathfrak{G}_{\lambda-1} \subseteq \mathfrak{A}_\nu$, and consequently

$$\mathfrak{G} = \mathfrak{A}_1 \mathfrak{A}_2 \dots \mathfrak{A}_\mu$$

and all factors are totally permutable. We have obtained a contradiction. The theorem is proved.

Definition 4. Let h be some divisor of the number $|\mathfrak{G}|$. Then a subgroup $\mathfrak{H}$ whose order is divisible by the largest $\pi(h)$ -divisor of the number $|\mathfrak{G}|$ will be called an h -subgroup of $\mathfrak{G}$.

Definition 5. If to each index h_i from $N_{\mathfrak{G}}$ there is assigned some h_i -subgroup $\mathfrak{H}_i$ of $\mathfrak{G}$, then the system of subgroups

$$\mathfrak{H}_i, \quad i = 1, 2, \dots, \lambda,$$

will be called an **invariant-index system of subgroups** of the group $\mathfrak{G}$ (in such a system of subgroups repetitions may occur).

Theorem 2. Every invariant-index system of subgroups of an arbitrary finite group $\mathfrak{G}$ is totally permutable and factorizes $\mathfrak{G}$.

Proof. We shall show that for an arbitrary invariant-index system of subgroups

$$\mathfrak{H}_i, \quad i = 1, 2, \dots, \lambda,$$

the conditions of Theorem 1 are fulfilled.

For this, first put $M_i = \{h_i\}$. We now verify the fulfillment of condition (2). By definition, the order $\mathfrak{H}_i$ is divisible by the largest $\pi(h_i)$ -divisor $\tilde{h}_i$ of the number $|\mathfrak{G}|$. But then

the largest $\pi(h_i)$ -divisors of $|\mathfrak{H}_i \cap \mathfrak{G}_j|$ and $|\mathfrak{G}_j|$ are equal to $\delta_{ij} = (\tilde{h}_i, |\mathfrak{G}_j|)$, $i = 1, 2, \dots, \lambda; j = 0, 1, \dots, \lambda$.

Assuming, as in Theorem 1, that

$$D_{ij} = M_i \cap H_j, \quad i = 1, 2, \dots, \lambda; j = 0, 1, \dots, \lambda,$$

we see that for $i < j + 1$ the set D_{ij} is empty and $\bar{D}_{ij} = 1$, and for

$j + 1 \leq i \leq \lambda$, $D_{ij} = \{h_i\}$ and $\bar{D}_{ij} = h_i$. From (3), in view of $|\mathfrak{G}_j| = \bar{H}_j$ and the divisibility of $\bar{h}_i$ by h_i , it follows that for $j + 1 \leq i \leq \lambda$ the number $|\mathfrak{H}_i \cap \mathfrak{G}_j|$ is divisible by $h_i = \bar{D}_{ij}$. Everything said shows that one may put

$$|\mathfrak{H}_i \cap \mathfrak{G}_j| = \bar{D}_{ij} d_{ij}, \quad \text{where } \bar{D}_{ij} = 1 \text{ for } i < j + 1 \text{ and } \bar{D}_{ij} = h_i \text{ for } j + 1 \leq i \leq \lambda, \text{ and } d_{ij} \text{ are certain factors.} \quad (4)$$

We shall prove that if $i = 1, 2, \dots, \lambda$ and $j = 0, 1, \dots, \lambda - 1$, then d_{ij} is divisible by d_{ij+1} . Obviously,

$$|\mathfrak{H}_i \cap \mathfrak{G}_j| = k_{ij} |\mathfrak{H}_i \cap \mathfrak{G}_{j+1}|, \quad i = 1, 2, \dots, \lambda; j = 0, 1, \dots, \lambda - 1. \quad (5)$$

Further, only the following cases are possible:

- 1) $i < j+1$. Then also $i < j+2$. Then from (4) it follows that $|\mathfrak{H}_i \cap \mathfrak{G}_j| = d_{ij}$, $|\mathfrak{H}_i \cap \mathfrak{G}_{j+1}| = d_{ij+1}$. Hence, by (5), we obtain the required assertion.
- 2) $i = j+1$. Then from (4) it follows that $|\mathfrak{H}_i \cap \mathfrak{G}_j| = h_i d_{ij}$ and $|\mathfrak{H}_i \cap \mathfrak{G}_{j+1}| = d_{ij+1}$. The index of $\mathfrak{G}_{j+1}$ in $\mathfrak{G}_j$ is h_i , which divides the greatest $\pi(h_i)$ -divisor of the number $|\mathfrak{G}_j|$, which by (3) is equal to δ_{ij} .

Consequently, $\delta_{ij} = h_i \delta_{ij+1}$. This equality, taking (3) into account, shows that the greatest $\pi(h_i)$ -divisor δ_{ij} of the left-hand side of (5) is obtained from the greatest $\pi(h_i)$ -divisor δ_{ij+1} of the number $|\mathfrak{H}_i \cap \mathfrak{G}_{j+1}|$ by multiplication by h_i . Consequently, the number k_{ij} in (5) must be divisible by h_i , i.e. $|\mathfrak{H}_i \cap \mathfrak{G}_j| = k'_{ij} h_i |\mathfrak{H}_i \cap \mathfrak{G}_{j+1}|$, or $h_i d_{ij} = k'_{ij} h_i d_{ij+1}$, whence $d_{ij} = k'_{ij} d_{ij+1}$, as was required to prove.

- 3) $i > j+1$. Then from (4) it follows that $|\mathfrak{H}_i \cap \mathfrak{G}_j| = h_i d_{ij}$ and $|\mathfrak{H}_i \cap \mathfrak{G}_{j+1}| = h_i d_{ij+1}$. Hence, on the basis of (5), the required assertion follows.

Thus, Theorem 1 may be applied. Theorem 2 is proved.

Theorem 3. *Let there be given some system $b_1, b_2, \dots, b_\nu$ of integral divisors of the order of an arbitrary finite group $\mathfrak{G}$ such that*

$$b_1 b_2 \dots b_\nu = |\mathfrak{G}|.$$

Then every system of subgroups $\mathfrak{B}_1, \mathfrak{B}_2, \dots, \mathfrak{B}_\nu$, whose orders are respectively equal to $b_1 d_1, b_2 d_2, \dots, b_\nu d_\nu$, is totally permutable and factorizes the group $\mathfrak{G}$.

Proof. As the series (1), take some chief series of $\mathfrak{G}$. According to the definition of a composition block and an integral-block divisor of a finite group, for each index h_i of the series (1) there is some subgroup of the system $\mathfrak{B}_1, \mathfrak{B}_2, \dots, \mathfrak{B}_\nu$ whose order is divisible by the greatest $\pi(h_i)$ -divisor of the number $|\mathfrak{G}|$. Assign it to the index h_i . It is now obvious that for the system $\mathfrak{B}_1, \mathfrak{B}_2, \dots, \mathfrak{B}_\nu$ all the conditions of Theorem 2 are satisfied, which proves Theorem 3.

Definition 6. Any collection of subgroups $\mathfrak{M}_1, \mathfrak{M}_2, \dots, \mathfrak{M}_k$, respectively of orders $m_1, m_2, \dots, m_k$, of an arbitrary finite group $\mathfrak{G}$, where $m_1, m_2, \dots, m_k$ are composition blocks of $\mathfrak{G}$, will be called a **composition collection of subgroups** of the group $\mathfrak{G}$.

Theorem 4. *Every composition collection of subgroups of an arbitrary finite group $\mathfrak{G}$ is totally permutable and factorizes $\mathfrak{G}$.*

Proof. In Theorem 3 it suffices to put $\nu = k$, $b_i = m_i$, $d_i = 1$, $i = 1, 2, \dots, k$, and Theorem 4 is proved.*

On the basis of our theorem mentioned in § 1, the systems of subgroups required by Theorems 3 and 4 always exist.

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Received
12 III 1969

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* The special case of Theorem 4 when $\mathfrak{G}$ is solvable was obtained by B. V. Kazachkov (his oral communication).

Note: Figure translations are in progress. See original paper for figures.

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