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Abstract

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MATHEMATICS

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ON BOUNDARY-VALUE PROBLEMS FOR ELLIPTIC-PARABOLIC EQUATIONS

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In this paper* boundary-value problems are studied for equations of the form

$$\mathfrak{A}(u) \equiv P_s(t, x, \partial/\partial t)u + L(t, x, D)u = h(t, x), \quad (1)$$

where

$$P_s\left(t, x, \frac{\partial}{\partial t}\right)u = \sum_{q=0}^s a_q(t, x)u^{(q)}, \quad u^{(q)} = \frac{\partial^q u}{\partial t^q}, \quad s \geq 1;$$

$$L(t, x, D)u \equiv \sum_{|\alpha| \leq 2m} a_\alpha(t, x)D^\alpha u$$

is an elliptic differential operator of order $2m$. The coefficients $a_q(t, x)$ and $a_\alpha(t, x)$ are smooth complex functions, $a_s = \pm 1$. In papers ^(2,3) we considered boundary-value problems for equations of the form (1) in a half-infinite cylinder, and also the problem of periodic solutions with respect to t . Here boundary-value problems for equation (1) are studied in the bounded cylinder $Q = G \times [0, T]$, where $G \subset \mathbf{R}^n$ is a domain with smooth boundary.

Definition. The operator $\mathfrak{A}(u)$ is called **elliptic-parabolic** (e.-p.) if, for all real $\tau \in \mathbf{R}^1$ and $\xi \in \mathbf{R}^n$ such that $|\tau| + |\xi| \neq 0$,

$$a_s(i\tau)^s + L_{2m}(t, x, i\xi) \neq 0,$$

where $L_{2m}(t, x, i\xi)$ is the principal part of the polynomial $L(t, x, i\xi)$.

For e.-p. equations the localization principle is valid. Therefore these equations are first studied in the unbounded cylinder $Q = G \times (-\infty, +\infty)$, then in the

cylinder $Q = G \times [0, \infty)$, and finally, by means of a partition of unity, in $Q = G \times [0, T]$.

§ 1. **The case of a self-adjoint operator.** In this paragraph $L(t, x, D) \equiv L(x, D)$. Suppose there is a system of boundary conditions

$$B_j(x, D)u|_{\partial G} = 0, \quad j = 1, \dots, m, \quad (2)$$

where

$$B_j(x, D)u \equiv \sum_{|\beta| \leq m_j} b_{\beta j}(x) D^\beta u$$

are differential operators of orders $m_j \leq 2m - 1$, defining together with the operator $L(x, D)u$ a self-adjoint boundary-value problem. Denote by $H^l\{B_j\}$ the space of Fourier series in the eigenfunctions $\omega_0(x), \omega_1(x), \dots$ of this problem, converging in the metric of H^l (see ^(4,5)).

Next, denote by $H(r, l)$ the space of functions $u(t, x)$ defined in the cylinder $Q = G \times (a, b)$, with norm

$$\|u\|_{r,l}^2 = \int_a^b (\|u^{(r)}\|_0^2 + \|u\|_l^2) dt. \quad (3)$$

The space of Fourier series

$$u(t, x) = c_0(t)\omega_0(x) + c_1(t)\omega_1(x) + \dots,$$

converging in $H(r, l)$, will be denoted by $H(r, l)\{B_j\}$.

* A detailed exposition is being published in the journal "Izv. AN Armenian SSR, Ser. Mathematics."

Theorem 1. Let $Q = G \times \mathbf{R}^1$, $a_q = 0$, $0 \leq q \leq s - 1$. If the operator $\mathfrak{A}(u)$ is e.p. and the first eigenvalue λ_0 of the operator $L(x, D)u$ is positive, i.e. $\lambda_0 > 0$, then it maps the space $H(r, r2m/s)\{B_j\}$ isomorphically onto the space $H(r - s, (r - s)2m/s)\{B_j\}$. Moreover,

$$\|h\|_{r-s, (r-s)2m/s} \leq C_1 (\|u\|_{r, r2m/s} + \lambda_0^{r/s} \|u\|_{0,0}) \leq C_2 \|h\|_{r-s, (r-s)2m/s}.$$

We now consider equation (1) in the semi-infinite cylinder $Q = G \times [0, \infty)$. It is required to find a solution of equation (1) under the initial conditions

$$u^{(n_i)}(0, x) = \psi_i(x), \quad i = 0, \dots, s - 1 - k, \quad (4)$$

where $k = [s/2]$, if s is odd and $(-1)^{[s/2]}a_s > 0$; $k = [s/2] + 1$, if s is odd and $(-1)^{[s/2]}a_s < 0$; $k = s/2$, if s is even.

Assume that for every $\lambda > 0$ the determinant

$$\det \|\mu_j^{n_i}(\lambda)\| \neq 0, \quad i, j = 0, \dots, s-1-k, \quad (5)$$

where $\mu_j(\lambda)$ are the roots of the algebraic equation $a_s\mu^s + \lambda = 0$ such that $\operatorname{Re} \mu_j(\lambda) < 0$.

Theorem 2. Let $Q = G \times [0, \infty)$, $a_q = 0$, $0 \leq q \leq s-1$; let the operator $\mathfrak{A}(u)$ be e.-p. and let condition (5) be satisfied. Then, if the first eigenvalue λ_0 of the operator $L(x, D)u$ is positive, i.e. $\lambda_0 > 0$, then for any function $h(t, x) \in H(r-s, (r-s)2m/s)\{B_j\}$ and any functions $\psi_i(x) \in H^{l_i}\{B_j\}$, $l_i = (r-n_i)2m/s - m/s$, there exists a unique solution $u(t, x) \in H(r, r2m/s)\{B_j\}$ satisfying conditions (2), (4). Moreover,

$$A(h, \psi) \leq C_1 \left(\|u\|_{r, r2m/s} + \lambda_0^{r/s} \|u\|_{0,0} \right) \leq C_2 A(h, \psi),$$

where

$$A(h, \psi) \equiv \|h\|_{r-s, (r-s)2m/s} + \sum_{i=0}^{s-1-k} \|\psi_i\|_{l_i}, \quad r \geq \max(2m, n_i + 1).$$

We now consider equation (1) with variable coefficients $a_q(t, x)$ in the bounded cylinder $Q = G \times [0, T]$. It is required to find a solution under the additional conditions

$$\sum_{q=0}^{n_i} p_{iq}(t, x) u^{(q)}|_{t=0} = \psi_i(x), \quad i = 0, \dots, s-1-k, \quad p_{in_i} \equiv 1; \quad (6)$$

$$\sum_{q=0}^{n_l} r_{lq}(t, x) u^{(q)}|_{t=T} = \chi_l(x), \quad l = 0, \dots, k-1, \quad r_{ln_l} \equiv 1. \quad (7)$$

Let, for sufficiently large $\lambda > 0$, the following condition be satisfied.

Condition 1 (the Shapiro-Lopatinskii condition):

- 1) $\det \|\mu_j^{n_i}(\lambda)\| \neq 0$, $i, j = 0, \dots, s-1-k$, where $\mu_j(\lambda)$ are the roots of the equation $a_s\mu^s + \lambda = 0$ such that $\operatorname{Re} \mu_j(\lambda) < 0$;
- 2) $\det \|\mu_j^{n_l}(\lambda)\| \neq 0$, $l, j = 0, \dots, k-1$, where $\mu_j(\lambda)$ are the roots of the equation $a_s\mu^s + \lambda = 0$ such that $\operatorname{Re} \mu_j(\lambda) > 0$.

Theorem 3. Let condition 1 be satisfied and let $\lambda_0 > 0$ be sufficiently large. Then problem (1), (2), (6), (7) is uniquely solvable, i.e. for any functions $h(t, x) \in H(r-s, (r-s)2m/s)\{B_j\}$, $\psi_i(x) \in H^{(r-n_i)2m/s-m/s}\{B_j\}$, $\chi_l(x) \in H^{(r-n_l)2m/s-m/s}\{B_j\}$, there exists a unique solution of this problem $u(t, x) \in H(r, r2m/s)\{B_j\}$. Moreover, the estimate

$$A(h, \psi, \chi) \leq C_1 \left(\|u\|_{r, r2m/s} + \lambda_0^{r/s} \|u\|_{0,0} \right) \leq C_2 A(h, \psi, \chi),$$

where

$$A(h, \psi, \chi) \equiv \|h\|_{r-s, (r-s)2m/s} + \sum_{i=0}^{s-1-k} \|\psi_i\|_{(r-n_i)2m/s-m/s} + \sum_{l=0}^{k-1} \|\chi_l\|_{(r-n_l)2m/s-m/s}.$$

Corollary. If condition 1 is satisfied, then problem (1), (2), (6), (7) is Fredholm, i.e., the kernel and cokernel of the problem are finite-dimensional, and their dimensions are equal.

§ 2. The case of the semibounded operator $L(t, x, D)u$

Suppose that the following conditions are satisfied:

I. For every fixed $t \in [0, T]$ the operator $L(t, x, D)u$ is bounded from below, i.e.,

$$\lambda_0(t) = \inf_{u \in H^{2m}\{B_j\}} \frac{\operatorname{Re} \langle L(t, x, D)u, \bar{u} \rangle}{\langle u, \bar{u} \rangle} > -\infty.$$

II. For any fixed $t \in [0, \varepsilon]$ ($t \in [T - \varepsilon, T]$), where $\varepsilon > 0$ is some number, the operator $L(t, x, D)u \equiv L(0, x, D)u$ ($L(t, x, D)u \equiv L(T, x, D)u$) is a self-adjoint operator in the space $H^{2m}\{B_j\}$.

III. Condition 1 of § 1 is satisfied.

IV. The coefficients of the operator $\mathfrak{A}(u)$ and of the boundary operators (6), (7) are sufficiently smooth, and moreover $a_q(t, x) \equiv a_q(t)$, $t \in [0, \varepsilon] \cup [T - \varepsilon, T]$.

Before formulating the main theorem of this paragraph, we introduce the necessary spaces. Denote by $\hat{H}(r, r2m/s)$ the space of functions $u(t, x) \in H(r, r2m/s)$ satisfying conditions (2) and such that, if $\varphi(t) \in C^\infty[0, T]$ and $\operatorname{supp} \varphi(t) \cap [0, \varepsilon] \neq \emptyset$ or $\operatorname{supp} \varphi(t) \cap [T - \varepsilon, T] \neq \emptyset$, then $\varphi(t)u(t, x) \in H(r, r2m/s)\{B_j\}$. In other words, it is required of functions $u(t, x) \in \hat{H}(r, r2m/s)$ that in some neighborhoods of the points $t = 0$ and $t = T$ they be expandable in the eigenfunctions of the operators $L(0, x, D)$ and $L(T, x, D)$, respectively. Otherwise they are arbitrary.

Theorem 4. If conditions I–IV are satisfied, then problem (1), (2), (6), (7) is Fredholm. This means that for any functions $h(t, x) \in \hat{H}(r-s, (r-s)2m/s)$,

$\psi_i(x) \in H^{(r-n_i)2m/s-m/s}\{B_j\}$, $i = 0, \dots, s-1-k$, $\chi_l(x) \in H^{(r-n_l)2m/s-m/s}\{B_j\}$, $l = 0, \dots, k-1$, subject to a finite number $\varkappa$ of conditions, there exists a solution of this problem $u(t, x) \in \hat{H}(r, 2m/s)$. At the same time it is determined up to the addition of any solution of the homogeneous problem, among which there are exactly $\varkappa$ linearly independent ones. If $\lambda_0 = \min \lambda_0(t)$, $0 \leq t \leq T$, is sufficiently large, then $\varkappa = 0$, i.e., there is an isomorphism. Moreover, the corresponding estimates hold.

In proving this theorem, estimates for elliptic problems with a parameter, obtained in [1], are used.

§ 3. Analogue of the Dirichlet problem. Uniqueness theorem. Examples

In this paragraph we consider the simplest example of problem (1), (2), (6), (7). Namely, let

$$\mathfrak{A}(u) \equiv P_s \left(\frac{\partial}{\partial t} \right) u + L(x, D)u = h(t, x), \quad (8)$$

where

$$P_s \left(\frac{\partial}{\partial t} \right) u \equiv \sum_{q=0}^s a_q u^{(q)}, \quad a_q \in C^1, \quad a_s = \pm 1,$$

$L(x, D)u$ is a semibounded operator of order $2m$.

We seek a solution of equation (8) under conditions (2), (9),

$$u(0, x) = \dots = u^{(s-1-k)}(0, x) = u(T, x) = \dots = u^{(k-1)}(T, x) = 0. \quad (9)$$

As follows from Theorem 4, problem (8), (2), (9) is Fredholm. We shall now be interested in the question of the unique solvability of this problem.

Theorem 5 (uniqueness). If the polynomial in $\tau \in \mathbb{R}^1$

$$\operatorname{Re} \sum_{q=0}^s a_q (i\tau)^q + \lambda_0 > 0,$$

then problem (8), (2), (9) has a unique solution.

Example 1. Consider the equation

$$a_3 u''' + (-1)^m \Delta^m u = h(t, x).$$

It is required to find a solution of this equation under conditions (2) such that

$$u(0, x) = u'(0, x) = u(T, x) = 0, \quad \text{if } a_3 = -1.$$

$$u(0, x) = u(T, x) = u'(T, x) = 0, \quad \text{if } a_3 = 1.$$

The problem has a unique solution.

Example 2. Consider the equation of even order

$$u^{(2k)} + (-1)^{m-k} \Delta^m u = h(t, x).$$

For this equation, the problem under investigation is the Dirichlet problem (in t):

$$u(0, x) = \dots = u^{(k-1)}(0, x) = u(T, x) = \dots = u^{(k-1)}(T, x) = 0. \quad (10)$$

By Theorems 4 and 5, the Dirichlet problem (10) is well posed. For an analogue of the Dirichlet problem, see ^(6, 7).

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