

A PRIORI ESTIMATES IN $(\mathscr{L}_2)$ OF SOLUTIONS OF GENERAL BOUNDARY-VALUE PROBLEMS FOR DEGENERATE ELLIPTIC EQUATIONS OF SECOND ORDER

V. P. Glushko

1969

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196901.30816>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.946

MATHEMATICS

V. P. GLUSHKO

A PRIORI ESTIMATES IN $\mathcal{L}_2$ OF SOLUTIONS OF GENERAL BOUNDARY-VALUE PROBLEMS FOR DEGENERATE ELLIPTIC EQUATIONS OF SECOND ORDER

(Presented by Academician I. G. Petrovskii on January 27, 1969)

The results obtained in ⁽¹⁾ are extended here to a broad class of degenerate elliptic operators of second order, defined in a domain D with smooth boundary.

Let D be a bounded domain of the Euclidean space E_n , and

$$S = \bigcup_{q=1}^N S_q$$

the boundary of D , consisting of closed nonintersecting $(n-1)$ -dimensional manifolds S_q (components).

Condition 1. Each component S_q of the boundary S belongs to the class C^m ($m \geq 2$). This means, in particular, that for each point $x^0 \in S_q$ there is a neighborhood $A_{x^0}(\delta)$ and a transformation $T_{x^0}(x) = (t, y)$ ($y \in E_{n-1}$) such that the intersection $A_{x^0}(\delta) \cap D$ is mapped by the transformation T_{x^0} into the half-ball $K_\delta : t^2 + |y|^2 \leq \delta^2, t > 0$.

We choose the transformation $T_{x^0}(x)$ in a special way so that, in the new local coordinates (t, y) , for points $x \in A_{x^0}(\delta) \cap D$, t is the coordinate of x along the inward normal to S , and $(y_1, y_2, \dots, y_{n-1}) = y$ are the local coordinates on S of the point of S through which this normal passes.

With such a choice of the transformation $T(x)$, the coordinates t of a point x in two local coordinate systems corresponding to the neighborhoods $A_{x^0}(\delta_0)$ and $A_{x^1}(\delta_1)$ ($x^0, x^1 \in S_q$) will coincide on the intersection of these neighborhoods. Thus, in some d -neighborhood of S_q a function $t = t(x) \in C^m$ will be defined. Together with it, on the intersection of the d -neighborhood with D , the function

$$a(x) = \sum_{i,j=1}^n a_{ij}(x) \partial_{x_i} t \partial_{x_j} t, \quad (1)$$

will be defined, where $a_{ij}(x)$ are the coefficients of the highest derivatives of the differential operator defined in D ,

$$\mathcal{L} = \sum_{i,j=1}^n a_{ij}(x) \partial_{x_i} \partial_{x_j} + \sum_{i=1}^n a_i(x) \partial_{x_i} + a_0(x).$$

Condition 2. The coefficients $a_{ij}(x), a_i(x)$ ($1 \leq i, j \leq n$) of the operator are real and continuous in $\overline{D}$. The operator $\mathcal{L}$ is elliptic at every interior point $x \in D$. For definiteness we shall assume that the characteristic polynomial

$$\sigma(x, \lambda) = \sum_{i,j=1}^n a_{ij}(x) \lambda_i \lambda_j > 0$$

for $x \in D$ and $\lambda \neq 0$. The boundary manifolds S_q are characteristic for $\mathcal{L}$ ($\sigma(x^0, \nu) = 0$ for $x^0 \in S_q$, ν being the inward normal to S at the point x^0).

Considering the operator $\mathcal{L}$ in $A_{x^0}(\delta) \cap D$, after the transformation $T_{x^0}(x) = (t, y)$ we can write it in the form

$$L_0 = \partial_t(a_{nn}^0(t, y) \partial_t) + \sum_{k=1}^{n-1} a_{nk}^0(t, y) \partial_t \partial_{y_k} + \sum_{k,l=1}^{n-1} a_{kl}^0(t, y) \partial_{y_k} \partial_{y_l} + a_n^0(t, y) \partial_t + \sum_{k=1}^{n-1} a_k^0(t, y) \partial_{y_k} + a_0^0(t, y).$$

Obviously,

$$a_{nn}^0 = a(T_{x^0}^{-1}(t, y)), \quad a_{nn}^0(t, y) > 0 \quad \text{for } t > 0 \quad \text{and} \quad a_{nn}^0(0, y) \equiv 0.$$

Condition 3. For every point $x^0 \in S_q$ the limits exist

$$\lim_{(t,y) \rightarrow (+0,0)} \frac{a_{nn}^0(t, y)}{a_{nn}^0(t, 0)} = 1; \quad (2)$$

$$\lim_{(t,y) \rightarrow (+0,0)} \frac{a_{nk}^0(t, y)}{\sqrt{a_{nn}^0(t, 0)}} = \gamma_k \quad (1 \leq k \leq n-1); \quad (3)$$

$$\lim_{(t,y) \rightarrow (+0,0)} \frac{|\text{grad}_y a_{nn}^0(t, y)|}{\sqrt{a_{nn}^0(t, 0)}} = 0.$$

Introducing the function $\alpha(t) = \sqrt{a_{nn}^0(t, 0)}$ and denoting $\alpha \partial_t = \alpha(t) \partial / \partial t$, we write L_0 in the form $L_0 = L'_0 + L''_0$, where

$$L'_0 = \frac{a_{nn}^0(t, y)}{\alpha^2(t)} (\alpha \partial_t)^2 + \sum_{k=1}^{n-1} \frac{a_{nk}^0(t, y)}{\alpha(t)} \alpha \partial_t \partial_{y_k} + \sum_{k,l=1}^{n-1} a_{kl}^0(t, y) \partial_{y_k} \partial_{y_l},$$

and L_0'' contains derivatives of order < 2 .

Definition. We shall say that the operator $\mathcal{L}$ is α -elliptic in $\bar{D}$ if it is elliptic in D and if at every point x^0 of the boundary characteristic manifold S_q ($q = 1, 2, \dots, N$), under conditions (2), (3), the quadratic form in $\eta, \xi = (\xi_1, \xi_2, \dots, \xi_{n-1})$

$$\zeta(\eta, \xi) = \eta^2 + \sum_{k=1}^{n-1} \gamma_k \eta \xi_k + \sum_{k,l=1}^{n-1} a_{kl}^0(0, 0) \xi_k \xi_l \neq 0 \quad (4)$$

does not vanish for any real $(\eta, \xi) \neq 0$. Since we have agreed to assume $\sigma(x, \lambda) > 0$, it follows from (4) that $\zeta(\eta, \xi) > 0$ for $(\eta, \xi) \neq 0$.

Let a boundary differential operator $\mathcal{B}_q$ be given on S_q , which in the neighborhood $A_{x^0}(\delta)$ can be written in the form

$$B_0(-\sqrt{-1} \partial_y, \partial_t) = \sum_{s=0}^r \sum_{|\tau| \leq \rho_s} b_{\tau,s}(y) (-\sqrt{-1} \partial_y)^\tau \partial_t^s,$$

where $\tau = (\tau_1, \tau_2, \dots, \tau_{n-1})$, $|\tau| = \tau_1 + \tau_2 + \dots + \tau_{n-1}$, and $b_{\tau,s}(y)$ are sufficiently smooth complex-valued functions.

Definition. We shall say that the degenerate order of $\mathcal{B}_q$ at the point $x^0 \in S_q$ is equal to m^* , if the degree of the polynomial

$$\sum_{s=0}^r \sum_{|\tau| \leq \rho_s} b_{\tau,s}(0) \xi^\tau \eta^{2s}$$

is equal to m^* .

For every point $x^0 \in S_q$ define the functions $b_\mu(t) = a_n^0(t, 0) + \mu \partial_t a_{nn}^0(t, 0)$, where μ is a real parameter, and construct the homogeneous

polynomial in ξ of degree m^*

$$\vartheta_{x^0}(\xi) = \sum_{s=1}^r \frac{b_0^s(0)}{b_1(0)b_2(0) \dots b_s(0)} \Lambda_s(\xi) \left(-\frac{c(\xi)}{b_0(0)} \right)^s + \Lambda_0(\xi),$$

where

$$\Lambda_s(\xi) = \sum_{|\tau|=m^*-2s} b_{\tau,s}(0) \xi^\tau, \quad c(\xi) = - \sum_{k,l=1}^{n-1} a_{kl}^0(0, 0) \xi_k \xi_l.$$

Definition. We shall say that the boundary operator $\mathcal{B}_q$ of degenerate order m^* satisfies at the point $x^0 \in S_q$ the condition of complementarity with respect to $\mathcal{L}$, if for any $\xi \neq 0$, $\vartheta_{x^0}(\xi) \neq 0$.

Condition 4. We shall assume that for the given integer $m \geq 2$ the components S_q of the boundary S can be divided into three sets (respectively $Q_1, Q_2(m)$, and $Q_3(m)$) according to the following rule:

- I. $q \in Q_1$, if $b_0(0) > 0$ for every point $x^0 \in S_q$.
- II. $q \in Q_2(m)$, if $b_{\frac{1}{2}(m-1)}(0) < 0$ for every point $x^0 \in S_q$.
- III. $q \in Q_3(m)$, if $b_0(0) < 0$ and there exists an integer $\tilde{m}_q \geq 1$ such that $b_{\frac{1}{2}(\tilde{m}_q-1)}(0) < 0$, $b_{\frac{1}{2}\tilde{m}_q}(0) \geq 0$ for all $x^0 \in S_q$, and moreover $m \geq \tilde{m}_q + 2$ if $\tilde{m}_q$ is even, and $m \geq \tilde{m}_q + 3$ if $\tilde{m}_q$ is odd.

As shown in ^(2,3), one can give a criterion for membership in $Q_1, Q_2(m), Q_3(m)$ that does not depend on the choice of a particular coordinate system at the point x^0 . We note that any of the sets $Q_1, Q_2(m), Q_3(m)$ may be empty.

Condition 5. Let, for an integer $m \geq 2$, the coefficients $a_{ij}(x)$ belong to $C^{m-1}(\overline{D})$; $a_i(x)$ ($0 \leq i \leq n$) belong to $C^{m-2}(\overline{D})$; the coefficients of the boundary operators $\mathcal{B}_q$ (of degenerate order m_q^*) belong to $C^{m-m_q^*}$ (where they are defined).

On the set $C^m(K_d)$ define the norm

$$\|v\|_{m,a}^2 = \sum_{r+2s+|\tau| \leq m} \int_{K_d} |a^i(t,y) \partial_t^{i+s} \partial_y^\tau v(t,y)|^2 dt dy,$$

where $a = a(t,y)$ is a weight function.

Construct a finite covering of $\overline{D}$, consisting of a strictly interior subdomain D_{δ_0} of the domain D and a system of neighborhoods $A_{x^p}(\delta_p)$, where $x^p \in S$, $p = 1, 2, \dots, P$. Let $\varphi_p(x)$ ($p = 0, 1, 2, \dots, P$) be functions belonging to $C_0^\infty(E_n)$ and forming a partition of unity corresponding to this covering,

$$\sum_{p=0}^P \varphi_p(x) = 1 \quad \text{in } \overline{D}, \quad \text{supp } \varphi_0 \subset D_{\delta_0}, \quad \text{supp } \varphi_p \subset A_{x^p}(\delta_p).$$

Definition. Let $m \geq 1$ be an integer. We shall say that a function $v(x) \in \mathcal{L}_2(D)$, having generalized derivatives in D up to order m , belongs to $H_a^m(D)$, if there exists such a covering $\overline{D} \setminus \{D_{\delta_0}, A_{x^p}(\delta_p), p = 1, 2, \dots, P\}$ that the norm

$$\|v\|_{m,a} = \left\{ \sum_{|\beta| \leq m} \int_D |\partial_x^\beta (\varphi_0 v)|^2 dx + \sum_{p=1}^P \|(\varphi_p v)_p\|_{m,a_p}^2 \right\},$$

is finite, where $(v)_p = v(T_{x^p}^{-1}(t, y))$; $a_p(t, y) = \sqrt{(a)_p}$, and the function $a(x)$ is defined by formula (1).

In the usual way we introduce the boundary norms $\langle \cdot \rangle_{S_q}^{q,m}$ on the manifolds S_q , putting for a finite function $v(y)$ ($y \in E_{n-1}$)

$$\langle v \rangle_{E_{n-1}, m} = \int_{E_{n-1}} |\xi|^{2m} |\tilde{v}(\xi)|^2 d\xi,$$

where $\tilde{v}(\xi)$ is the Fourier transform in E_{n-1} of the function $v(y)$.

Theorem. Let, for a given integer $m \geq 2$, the operator $\mathcal{L}$ and the domain D with boundary

$$S = \bigcup_{q=1}^N S_q$$

satisfy conditions 1-5. On each boundary manifold S_q , for $q \in Q_2(m)$, let a boundary operator $\mathcal{B}_q$ be given, satisfying condition 5 and having on S_q a pronounced order equal to

$$m_q^* \leq 2 \left[\frac{m-2}{2} \right].$$

Then, for the inequality

$$\|v\|_{m,\alpha} \leq c_m \left\{ \|\mathcal{L}v\|_{m-2,\alpha} + \sum_{q \in Q_2(m)} \langle \mathcal{B}_q v \rangle_{S_q, m-m_q^*-1} + \|v\|_0 \right\}, \quad (5)$$

to hold, where $v(x)$ is any function in $H_\alpha^m(D)$, it is necessary and sufficient that the operator $\mathcal{L}$ be α -elliptic in $\bar{D}$ and that the boundary operators $\mathcal{B}_q$ satisfy the complementing condition with respect to $\mathcal{L}$ on S_q for $q \in Q_2(m)$.

One may consider the following boundary value problem: find a function

$$v(x) \in H_\alpha^{m-2}(D)$$

satisfying the conditions

$$\begin{aligned} \mathcal{L}v &= f \quad \text{in } D; \\ \mathcal{B}_q v &= g_q \quad \text{on } S_q \text{ for } q \in Q_2(m), \end{aligned} \quad (6)$$

where $f \in H_\alpha^{m-2}(D)$ and $g_q (\langle g_q \rangle_{S_q, m-m_q^*-1} < \infty)$ are prescribed functions.

If, for the given $m \geq 2$, the hypotheses of the theorem under which inequality (5) holds are fulfilled, and if $Q_3(m)$ is the empty set, then the operator corresponding to the boundary value problem (6) is Noetherian.

Voronezh State University

Received
18 I 1969

REFERENCES

1. V. P. Glushko, *Functional Analysis and Its Applications*, 2, 3, 1968, p. 87.
2. O. A. Oleinik, *Mat. sbornik*, 69 (111), 1 (1966).
3. J. J. Kohn, L. Nirenberg, *Comm. Pure and Appl. Math.*, 18, No. 3, 443 (1965); Russian transl.: J. J. Kohn, L. Nirenberg, in the collection *Pseudodifferential Operators*, Moscow, 1967, p. 88.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.