

# ON CERTAIN $\backslash(\backslash\Delta\backslash\Sigma\backslash)$ - OPERATIONS WHOSE BASES ARE OBTAINED BY UNITING THE CHAINS OF THE BASE OF THE ORIGINAL OPERATION

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**Abstract**

**Full Text**

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**MATHEMATICS**

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# ON CERTAIN $\Delta\Sigma$ -OPERATIONS WHOSE BASES ARE OBTAINED BY UNITING THE CHAINS OF THE BASE OF THE ORIGINAL OPERATION

*(Presented by Academician L. V. Kantorovich on 8 I 1969)*

Let  $I$  be the space of indices of cardinality  $\tau = \aleph_\nu$ ; let  $\tau$  be a strongly inaccessible cardinal number; let  $\mathfrak{N}$  be the basic space whose sets are under study; and let  $H = (H_\xi)$  be a family of bases possessing some property  $H$ . We denote by  $\overline{H}$  the set of all bases lacking the property  $H$ . If the space of chains  $\Xi$  is a topological space, then we shall call the property of bases  $H$  a topological property when the closure  $[H_\xi]$  has the same property as  $H_\xi$ .

If a base  $N$  is given, then we construct a new base

$$HN = \left( \bigcup_{\eta \in H_\xi} \eta \right)_{H_\xi \in H},$$

obtained by uniting chains of subsets  $H_\xi$  of the base  $N$  possessing the property  $H$ . If  $\Phi_{HN} \prec \Phi_N$ ,  $\Phi_{(HN)^i} \prec \Phi_N$  for  $i \in I$ , then the property of bases  $H$  will be called  $N$ -regular. We clarify the conditions of  $N$ -regularity of the following properties of bases:  $H_p$  (to contain at least  $p$  distinct chains of the base  $N$ ,  $2 \leq p < \omega$ ),  $H_{\tau'}$  (to contain at least  $\tau'$  distinct chains of the base  $N$ , where  $\tau' = \aleph_\nu \leq \tau$ ).

Let  $N$  be the base of some  $\Delta\Sigma$ -operation. The set of all chains of the base  $N$  containing the index  $i$  is called the truncated base  $N^i$ . Let  $\mathfrak{M} = \{N\} \cup \bigcup (N^i)$  and  $\tau^* \leq \aleph_{\nu+1}$ . Denote by  $\mathfrak{M}_{\tau^*}$  the family of all intersections of sets of the family  $\mathfrak{M}$  in number  $< \tau^*$ .

**Lemma 1.** *If  $K \neq \emptyset$  and the base  $N$  is such that for every  $M \in \mathfrak{M}_{\tau^*}$  the inclusion  $\Phi_M(K) \subset \Phi_N(K)$  holds, then*

$$\Phi_{M_1 \setminus M_2}(K) \subset \Phi_N(K), \quad \Phi_{M_1 \setminus \bigcup_{\alpha} M_1^{i_\alpha}}(K) \subset \Phi_N(K),$$

where  $M_1, M_2 \in \mathfrak{M}_{\tau^*}$ ,  $\alpha \in I$  (the last inclusion was observed by Yu. I. Romanovsky <sup>(4)</sup>).

We shall call the base of a  $\Delta\Sigma$ -operation  $N$   $\tau^*$ -regular if: 1)  $(\Phi_N, \bigcup_{\tau}) \prec \Phi_N$ ,  $(\Phi_N, \bigcap_{\tau}) \prec \Phi_N$ ; 2) if  $M \in \mathfrak{M}_{\tau^*}$ , then  $\Phi_M \prec \Phi_N$ . We shall call the base of a  $\Delta\Sigma$ -operation  $N$   $(\aleph_0)_{\nu}$ -semiregular for  $\nu < \omega_{\nu}$  if: 1)  $(\Phi_N, \bigcup_{\tau}) \prec \Phi_N$ ,  $(\Phi_N, \bigcap_{\text{card } \nu}) \prec \Phi_N$ ; 2) if  $M \in \mathfrak{M}_{\aleph_0}$ , then  $\Phi_M \prec \Phi_N$ .

**Lemma 2.** Let  $N$  and  $N^c$  be rigid bases of  $\Delta\Sigma$ -operations,  $\mathfrak{M} = \{N\} \cup \bigcup (N^i)$ . In order that the base  $M$  of the operation  $\Phi_M \equiv (\nu)T_{\{N\}}$  be  $\tau^*$ -regular for every  $\tau^* \leq \aleph_{\nu+1}$ , where  $\nu = \omega \cdot \lambda \leq \omega_{\nu}$ , it is sufficient that the following conditions be fulfilled: 1)  $((\nu)T_N, \bigcup_{\tau}) \prec (\nu)T_N$ ,  $((\nu)T_N, \bigcap_{\tau}) \prec (\nu)T_N$ ; 2) if  $L \in \mathfrak{M}_{\tau^*}$ , then  $\Phi_L \prec \Phi_N$ , or, if the base  $N$  satisfies condition 2<sup>0</sup>)  $(\Phi_N, d) \prec \Phi_N$ ,  $(\Phi_{N^c}, d) \prec \Phi_{N^c}$ , or 2')  $(\nu)T_N \prec (\nu)T_{N^c}$ ,  $((\nu)T_{N^c}, d) \prec (\nu)T_{N^c}$ , then  $\Phi_L \prec (\nu)T_N$ . In order that the base  $M$  be  $(\aleph_0)_{\nu}$ -semiregular for  $\nu < \omega_{\nu}$ , it is sufficient that the conditions be fulfilled: 1\*)  $((\nu)T_N, \bigcup_{\tau}) \prec (\nu)T_N$ ,  $((\nu)T_N, \bigcap_{\text{card } \nu}) \prec (\nu)T_N$ ; 2\*) if  $L \in \mathfrak{M}_{\aleph_0}$ , then  $\Phi_L \prec \Phi_N$ , or, if the base  $N$  satisfies condition 2<sup>0</sup>) or 2'), then  $\Phi_L \prec (\nu)T_N$ . In order that

in order that the rigid base  $M^c$  be  $\tau^*$ -regular with respect to a class  $K \supseteq \emptyset$  for every  $\tau^* \leq \aleph_{\nu+1}$ , it is sufficient that conditions 1) and 2\* be fulfilled with respect to this class (the conditions of  $\aleph_{\nu+1}$ -regularity of the bases  $M$  and  $M^c$  were proved by Yu. I. Romanovskii <sup>(4)</sup>).

**Theorem 1.** If the rigid base of the  $\Delta\Sigma$ -operation  $N$  is  $\aleph_0$ -regular or  $(\aleph_0)_{\omega}$ -semiregular with respect to the class of sets  $K \supseteq \emptyset$ , then the properties of the bases  $H_p$  ( $2 \leq p < \omega$ ),  $H_{\aleph_0}$  are  $N$ -regular.

**Corollary.** If  $\mathfrak{A}$  is the rigid base of a  $(\nu)A$ -operation, where  $\nu = \omega \cdot \lambda \leq \omega_{\nu}$ , then the properties of the bases  $H_p$  ( $2 \leq p < \omega$ ),  $H_{\aleph_0}$  are  $\mathfrak{A}$ -,  $\mathfrak{A}^c$ -regular.

**Theorem 2.** If  $N$  and  $N^c$  are rigid bases of  $\Delta\Sigma$ -operations, the rigid base  $M$  of the operation  $\Phi_M \equiv (\nu)T_{\{N\}}$  satisfies conditions 1) and 2) of Lemma 2, then the property of the bases  $H_{\tau'}$  is  $M$ -regular with respect to any class of sets  $K \supseteq \emptyset$ , where  $\tau' = \aleph_{\nu} \leq \tau$ .

Indeed, consider  $\tau'$  different chains of the base  $M$ , where  $\tau' < \tau$ . Since  $M$  is a rigid base, one can indicate  $\tau'$  different indices, pairwise not subordinate to one another, each of which enters one and only one of the chains under consideration. Such indices can be arranged in sequences of type  $\omega_{\nu}$ . Let  $i_0, \dots, i_{\alpha}, \dots \mid \omega_{\nu}$  be one of such sequences. Then

$$\Phi_{H_{\tau'} M} \{E_i\} = \bigcup_{i_0, \dots, i_{\alpha}, \dots \mid \omega_{\nu}} \bigcap_{M^{i_p}} \Phi_M^{i_p} \bigcup_{\alpha \neq p, \alpha < \omega_{\nu}} M^{i_{\alpha}} \{E_i\},$$

where the union is taken over all possible sequences of pairwise non-subordinate indices of type  $\omega_{\nu}$ , belonging to the set  $\zeta$  of all indices entering the chains of the base  $M$ , and  $(E_i)$  is a family of sets of the class  $K \supseteq \emptyset$ . Since the base  $M$ , by Lemma 2, is  $\tau^*$ -regular for every  $\tau^* \leq \aleph_{\nu+1}$ , it follows, by Lemma 1, that

$$\Phi_M^{i_p} \bigcup_{\alpha \neq p, \alpha < \omega_\nu} M^{i_\alpha} \{E_i\} \subseteq \Phi_M(K).$$

Since  $\tau$  is a strongly inaccessible cardinal number, the number of different sequences of pairwise non-subordinate indices  $i_0, \dots, i_\alpha, \dots \mid \omega_\nu$ , belonging to  $\zeta$ , is not greater than  $\tau^{\tau'} = \tau$ . By condition 1) of Lemma 2,  $\Phi_{H_{\tau'}M}(K) \subseteq \Phi_M(K)$ . The operation

$$\Phi_{H_\tau M} \{E_i\} = \bigcap_{\tau' < \tau} \Phi_{H_{\tau'} M} \{E_i\}.$$

By condition 1) of Lemma 2,  $\Phi_{H_\tau M}(K) \subseteq \Phi_M(K)$ . The operations

$$\Phi_{(H_{\tau'}M)^i} \{E_j\} = \Phi_{H_{\tau'}M} \{E_j\} \cap \Phi_{M^i} \{E_j\},$$

$$\Phi_{(H_\tau M)^i} \{E_j\} = \Phi_{H_\tau M} \{E_j\} \cap \Phi_{M^i} \{E_j\},$$

whence it follows that

$$\Phi_{H_{\tau'}M}(K) \subseteq \Phi_M(K), \quad \Phi_{H_\tau M}(K) \subseteq \Phi_M(K).$$

**Corollary.** The property of the bases  $H_{\tau'}$ , for  $\tau' = \aleph_\nu \leq \tau$ , is  $\mathfrak{A}$ -regular.

On the basis of the first and second theorems on multiple separation for the class of sets  $\Phi_N(K)$  with respect to the operation  $\Phi_{HN}$ , in the case when the property of the bases  $H$  is  $N$ -regular, we obtain the first and second theorems on coverings of sets.

**First theorem on coverings of sets.** If  $N$  is a rigid  $\tau^*$ -regular or  $(\aleph_0)_v$ -semiregular base, where  $v = \omega \cdot \lambda < \omega_\nu$ , the operation  $\Phi_N$  is normal, the class of sets  $\Phi_N(K)$  has a system of fully regular indices,  $H$  is an  $N$ -regular property of bases, then for every family of sets  $(E_i)$  of the class  $\Phi_N(K)$  such that  $\Phi_N \{E_i\} \subseteq \Phi_{HN} \{E_i\}$ ,

there will be found a family of sets  $(U_i)$  of the class  $B\Phi_N(K)$  such that  $U_i \supset E_i$  and  $\Phi_N \{U_i\} \subset \Phi_{HN} \{U_i\}$ .

**The second theorem on covering sets.** If  $N$  is a rigid  $\tau^*$ -regular base or an  $(\aleph_0)_\nu$ -semiregular base, where  $\nu = \omega \cdot \lambda < \omega_\nu$ , the operation  $\Phi_N$  is normal, the class of sets  $\Phi_N(K)$  possesses a system of regular transfinite indices, and  $H$  is an  $N$ -regular property of bases, then for every family of sets  $(E_i)$  of the class  $\Phi_N(K)$  there will be found a family of sets  $(U_i)$  of the class  $\Phi_{N^c}(K^c)$  such that

$$U_i \supset E_i \setminus \Phi_{(HN)^i} \{E_j\}, \quad \Phi_{HN} \{U_i\} = \phi.$$

These theorems hold for the properties  $H_p$  and  $\bar{H}_\tau$ , for  $\tau' = \aleph_{\nu'} \leq \tau$ , in the following cases:

- 1) for the operations  $\Phi_{N_{\alpha+1}} \equiv T_N^{\alpha+1}$  with depth of chains  $\omega$  relative to the class of sets  $T_N^{\alpha+1}(K)$ , where  $K \supset \mathfrak{M}$ ,  $N$  and  $N^c$  are rigid bases, the base  $N$  satisfies the conditions: 1<sup>0</sup>)  $T_N \succ \bigcup_{\aleph_0}$ ,  $T_N \succ \bigcap_\tau$ ,  $T_N \succ \Phi_{N^c}$ ; 2<sup>0</sup>)  $(\Phi_N, \tilde{\mathcal{A}}) \prec \Phi_N$ ,  $(\Phi_{N^c}, d) \prec \Phi_{N^c}$ ; 3<sup>0</sup>) if  $L \in \mathfrak{M}_{\tau^*}$ , where  $\mathfrak{M} = \{N\} \cup (N^i)$ , then  $\Phi_L \prec \Phi_N$  or  $\Phi_N \prec T_N$  (if  $N$  is a rigid base of the operation  $\bigcup_{\aleph_0}$ , then we obtain the results of A. A. Lyapunov <sup>(1)</sup> and of the author <sup>(3)</sup>);
- 2) for the  $(\omega_\nu)$   $H$ -operations  $\Phi_{\mathfrak{A}}$ .

A point  $x \in \mathfrak{A}$  is called a point of  $N$ - $p$ -valuedness (less than  $N$ - $\tau'$ -valuedness) of a family of sets  $(E_i)$ , if there exist  $p$  and only  $p$  distinct chains of the base  $N$  (less than  $\tau'$  distinct chains of the base  $N$ ) whose kernels contain the point  $x$ . For  $p = 1$  we obtain points of  $N$ -single-valuedness of the given family of sets; for  $\tau' = \aleph_0$ , points of  $N$ -finite-valuedness. We shall clarify the nature of these points for individual operations.

**Theorem 3.** If  $N$  is a rigid base of a  $\Delta\Sigma$ -operation satisfying conditions 1<sup>0</sup>, 2<sup>0</sup>, 3<sup>0</sup>;  $(E_i)$  is a family of sets of the class  $BT_N^{\alpha+1}(K)$ ;  $N_{\alpha+1}$  is a rigid base of an operation of type  $T_N^{\alpha+1}$ , then the set of points of  $N_{\alpha+1}$ -single-valuedness of the family of sets  $(E_i)$  belongs to the class of sets  $CT_N^{\alpha+1}(K)$ .

For  $\tau = \aleph_0$  the theorem was proved by A. A. Lyapunov <sup>(1)</sup>.

**Theorem 4.** If  $N$  is a rigid base of a  $\Delta\Sigma$ -operation satisfying conditions 1<sup>0</sup>, 2<sup>0</sup>, 3<sup>0</sup>;  $K \supset \phi$ ;  $(E_i)$  is a family of sets of the class  $BT_N^{\alpha+1}(K)$ ;  $N_{\alpha+1}$  is a rigid base of an operation of type  $T_N^{\alpha+1}$ , then the set of points of  $N_{\alpha+1}$ - $p$ -valuedness of the family of sets  $(E_i)$  belongs to the class  $CT_N^{\alpha+1}(K)$ .

For  $\tau = \aleph_0$  we have the theorem of A. A. Lyapunov <sup>(1)</sup>.

**Theorem 5.** If  $N$  and  $N^c$  are rigid bases, the base  $N$  has the properties 1<sup>0</sup>, 2<sup>0</sup>, 3<sup>0</sup>;  $N_{\alpha+2}$  is a rigid base of an operation of type  $T_N^{\alpha+2}$ ;  $(E_i)_i$  is a family of sets of the class  $BT_N^{\alpha+2}(K) [BT_N^1(K)]$ , where  $K \supset \phi, \mathfrak{A}$ , then the set of points of  $N_{\alpha+2}$ - $\tau'$ -valuedness ( $N_1$ - $\tau'$ -valuedness), for any  $\tau' = \aleph_{\nu'} < \tau$  and  $\alpha < \omega_{\nu+1}$ , belongs to the class  $CT_N^{\alpha+2}(K) [CT_N^1(K)]$ .

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*Note: Figure translations are in progress. See original paper for figures.*

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