

AN A PRIORI ESTIMATE OF THE PRINCIPAL RADII OF CURVATURE OF A CLOSED CONVEX HYPERSURFACE IN TERMS OF THEIR MEAN VALUES

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Abstract

Full Text

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MATHEMATICS

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AN A PRIORI ESTIMATE OF THE PRINCIPAL RADII OF CURVATURE OF A CLOSED CONVEX HYPERSURFACE IN TERMS OF THEIR MEAN VALUES

We consider a closed convex hypersurface F in $(n + 1)$ -dimensional Euclidean space, satisfying the equation

$$S_k(R_1, \dots, R_n) = \bar{S}_k(\nu), \quad (1)$$

where S_k is the elementary symmetric function of the principal radii of curvature of the hypersurface at a point with exterior normal ν , and $\bar{S}_k(\nu)$ is a prescribed function of the unit vector ν . The problem is to estimate the principal radii of curvature of the hypersurface in terms of the function $\bar{S}_k$. In the case $k = n$ such an estimate was obtained in the author's paper ⁽¹⁾.

Denote

$$\bar{\rho}_k(\nu) = \left(\frac{\bar{S}_k(\nu)}{C_n^k} \right)^{1/k}$$

and call the quantity $\bar{\rho}_k(\nu)$ the mean radius of normal curvature of the hypersurface F at the point with exterior normal ν .

Theorem. *For the radii of normal curvature of a closed convex hypersurface F the estimate*

$$R \leq \max_{X, \gamma} (\bar{\rho}_k(X) - \bar{\rho}_k''(X)),$$

holds, where the differentiation of $\bar{\rho}_k$ is performed along the arc of the great circle γ issuing from the point X on the unit sphere Ω , and the maximum is taken over all points X of the sphere and over all directions γ from this point.

Proof. Let H be the support function of the hypersurface F . Then the principal radii of curvature R_k are the roots of the polynomial

$$P(R) = \|H_{ij} + R\delta_{ij}\| = 0,$$

where H_{ij} are the second derivatives of H on the unit sphere. Put

$$h(x_1, x_2, \dots, x_n) = H(x_1, x_2, \dots, x_n, 1).$$

Then, taking into account the homogeneity of the function H , the derivatives H_{ij} can be expressed in terms of the derivatives of the function h , and the equation $P(R) = 0$ for the principal radii of curvature, after elementary transformations, is represented in the form

$$P(R) = \begin{vmatrix} \lambda h_{11} + R & \lambda h_{12} & \dots & x_1 R \\ \lambda h_{21} & \lambda h_{22} + R & \dots & x_2 R \\ \dots & \dots & \dots & \dots \\ x_1 R & x_2 R & \dots & R(1 + x_1^2 + \dots + x_n^2) \end{vmatrix} = 0, \quad (2)$$

where

$$h_{ij} = \partial^2 h / \partial x_i \partial x_j, \quad \lambda = (1 + x_1^2 + \dots + x_n^2)^{1/2}, \quad (3)$$

$$P(R) = R^{n+1} + S_1 R^n + \dots + S_n R.$$

The coefficients S_k of the polynomial $P(R)$ are elementary symmetric functions of the principal radii of curvature,

$$S_k(R_1 \dots R_n) = \sum_{i_\alpha \neq i_\beta} R_{i_1} R_{i_2} \dots R_{i_k}.$$

As was shown in ⁽¹⁾, one can introduce a coordinate system $x_1, \dots, x_n$ such that the function

$$w = h_{11}(1 + x_1^2 + \dots + x_n^2)^{3/2} / (1 + x_2^2 + \dots + x_n^2)$$

at the point $O : x_1 = x_2 = \dots = x_n = 0$ attains an absolute maximum, and this maximum is equal to the largest radius of normal curvature of the hypersurface F . Thus the problem reduces to estimating this maximum. At the point O

$$h_{ii} = R_i, \quad h_{ij} = 0 \text{ for } i \neq j; \quad w_i = h_{11i} = 0; \quad (4)$$

$$w_{11} = (h_{11})_{11} + 3R_1 \leq 0, \quad w_{ii} = (h_{11})_{ii} + R_1 \leq 0, \quad i > 1. \quad (5)$$

Let us compute the derivative $S'_k = dS_k/dx_1$ at the point O . It is the coefficient of the polynomial $P' = dP/dx_1$. We have

$$P' = \sum_i R(R_1 + R) \dots (R_{i-1} + R)(R_{i+1} + R) \dots (R_n + R)h'_{ii}.$$

Here the summation is over $i > 1$. But since $h'_{11} = 0$, it may be assumed that the summation extends over all values of i from 1 to n . The coefficient of R^{n-k} in the polynomial P' has the form

$$S'_{k+1} = \sum_i h'_{ii} S_k^i,$$

where S_k^i is the elementary symmetric function of the variables $R_1, \dots, R_{i-1}, R_{i+1}, \dots, R_n$. Noting that $S_k^i = \partial S_{k+1} / \partial R_i$, we obtain

$$S'_{k+1} = \sum_i h'_{ii} \frac{\partial S_{k+1}}{\partial R_i}.$$

Thus,

$$S'_k = dS_k, \quad \text{if } dR_i = h'_{ii}. \quad (6)$$

We now compute the second derivative S''_k with respect to the variable x_1 at the point O . For this we differentiate the polynomial P with respect to x_1 twice. Omitting the corresponding calculations, we give the value of the coefficient S''_k that interests us:

$$S''_k = S_{k-1}^i (h_{11})_{ii} + (k+2)S'_k - 2S'_k + 2S_{k-2}^{ij} (h'_{ii} h'_{jj} - h_{ij}^{'2}),$$

where S_{k-2}^{ij} is the elementary symmetric function of the variables $R_1, R_2, \dots$, except for R_i and R_j . For simplicity of notation, the summation sign will henceforth be omitted.

Taking into account the inequalities (5), after simple transformations we obtain

$$S''_k \leq -R_1(n-k+1)S_{k-1} + kS_k + 2S_{k-2}^{ij} (h'_{ii} h'_{jj} - h_{ij}^{'2}).$$

We strengthen this inequality by dropping the term $-h_{ij}^{'2}$. Then

$$S_k'' \leq -R_1(n-k+1)S_{k-1} + kS_k + 2S_{k-2}^{ij}h'_{ii}h'_{jj}, \quad (7)$$

Note that

$$S_{k-2}^{ij}h'_{ii}h'_{jj} = d^2S_k, \quad \text{if } dR_i = h'_{ii},$$

and use the concavity of the function $(S_k)^{1/k}$ in the variables R_i (2). We have

$$d^2(S_k^{1/k}) \leq 0.$$

Hence

$$d^2S_k \leq \left(1 - \frac{1}{k}\right) \frac{(dS_k)^2}{S_k} = \left(1 - \frac{1}{k}\right) \frac{(S_k')^2}{S_k}.$$

Now inequality (7) is strengthened as follows:

$$S_k'' \leq -R_1(n-k+1)S_{k-1} + kS_k + \left(1 - \frac{1}{k}\right) \frac{S_k'^2}{S_k}.$$

Hence

$$R_1 \leq \frac{kS_k + (1 - 1/k)S_k'^2/S_k - S_k''}{(n-k+1)S_{k-1}}. \quad (8)$$

Noting that

$$\left(\frac{S_{k-1}}{C_n^{k-1}}\right)^{1/(k-1)} \geq \left(\frac{S_k}{C_n^k}\right)^{1/k},$$

from (8) we finally obtain the required estimate

$$R \leq \left(\frac{S_k}{C_n^k}\right)^{1/k} - \left[\left(\frac{S_k}{C_n^k}\right)^{1/k}\right]''.$$

Differentiation with respect to x_1 at the point O may be replaced by differentiation along the arc of a great circle issuing in the direction of the axis x_1 . The theorem is proved.

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2. E. Beckenbach, R. Bellman, *Inequalities*, Moscow, 1965.

Note: Figure translations are in progress. See original paper for figures.

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