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Abstract

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MATHEMATICS

S. I. POKHOZHAEV

ON THE NORMAL SOLVABILITY OF NON-LINEAR EQUATIONS

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Let L be a linear bounded operator acting from a Banach space* X into a Banach space Y , and let $L^* : X^* \rightarrow Y^*$ be the operator adjoint to L .

Consider the equation

$$Lx = y. \quad (1)$$

If, for some $y \in Y$, equation (1) is solvable, then, as is known,

$$y \in (\text{Ker } L^*)^\perp \equiv \{y \in Y \mid \langle y^*, y \rangle = 0 \text{ for every } y^* \in \text{Ker } L^*\}.$$

Here $\langle y^*, y \rangle$ is the value of the functional $y^* \in Y^*$ on the element $y \in Y$, and

$$\text{Ker } L^* \equiv \{y^* \in Y^* \mid L^*y^* = 0\}.$$

The linear equation (1) is called normally solvable in the sense of Hausdorff (see ⁽¹⁾) if this necessary condition for solvability is also sufficient.

Let now $A(x)$ be a continuous and continuously differentiable (nonlinear) operator acting from X into Y . Consider the equation

$$A(x) = y. \quad (2)$$

If equation (2) is solvable for some $y \in Y$, i.e., there exists $x_0 \in X$ such that $A(x_0) = y$, then, obviously,

$$y - A(x_0) \in (\text{Ker}(A'(x_0))^*)^\perp.$$

Here $(A'(x_0))^*$ is the operator adjoint to the linear bounded operator $A'(x_0)$, the derivative of the operator $A(x)$ at $x = x_0$.

Definition. Equation (2) is called normally solvable if: 1) the operator $A(x)$ is of class C^1 and is such that for every $y \in Y$ there exists an x_0 , a point of minimum** of the functional $\|A(x) - y\|$; 2) from the condition

$$y - A(x_0) \in (\text{Ker}(A'(x_0))^*)^\perp$$

it follows that $y = A(x_0)$.

Remark. In the case of a reflexive Banach space Y and a linear bounded operator $A : X \rightarrow Y$, this definition is equivalent to Hausdorff's definition.

In this note a theorem is proved giving a sufficient condition for the normal solvability of equation (2). As a consequence, a theorem on the solvability of the nonlinear equation (2) is obtained. A class of operators $A(x)$ is described for which the "orthogonality" condition takes the form

$$y \in (\text{Ker}(A'(x_0))^*)^\perp.$$

An extension of the operator A is considered, based on the introduction of additional "potentials."

Theorem 1. *Let Y be a reflexive Banach space and let A be an operator of class C^1 acting from the Banach space X into Y . Let $A(X)$, the range of the operator A , be weakly closed*** in Y .*

Then the nonlinear equation (2) is normally solvable.

The proof is based on the following lemmas.

* Here and below all Banach spaces are considered over the field of real or complex numbers.

** Everywhere in the text, by a minimum is meant an absolute (i.e., relative to all of X) minimum.

*** Everywhere in the note, following S. Mazur, a weakly closed set is understood to mean a set that contains all weak limits of sequences of elements of this set.

Lemma 1. *Under the conditions of Theorem 1, for any $y \in Y$ the functional $f(x) \equiv \|A(x) - y\|$ attains its lower bound.*

Let x_0 be a point of minimum of the functional $f(x)$ for an arbitrarily fixed $y \in Y$. Introduce the notation: $A_0 = A'(x_0)$, RA_0 is the range of the linear operator A_0 , and $\overline{RA_0}$ is the closure of the set RA_0 , $v_0 = A(x_0) - y$.

Lemma 2. *Let $v_0 \in \overline{RA_0}$. Then $v_0 = 0$.*

Proof. Suppose that $v_0 \neq 0$. Since $v_0 \in \overline{RA_0}$, for $\delta = \|v_0\|/2 > 0$ there is an element $v_1 \in Y$ such that $\|v_1 - v_0\| < \delta$, and the equation $A_0 u = v_1$ has a solution. Take some solution u_0 of this equation (if the solution of this equation is not unique), $A_0 u_0 = v_1$, and fix it. Since $\|v_1\| > 0$, we have $\|u_0\| > 0$. Putting $x - x_0 = \lambda u_0$, we obtain $A_0(x - x_0) = \lambda v_1$ (λ is a real number).

We now consider the differentiability condition of the operator $A(x)$:

$$A(x) = A(x_0) + A_0(x - x_0) + \omega(x_0, x - x_0),$$

where

$$\|\omega(x_0, x - x_0)\|/\|x - x_0\| \rightarrow 0 \quad \text{as } \|x - x_0\| \rightarrow 0.$$

Since $f(x) \equiv \|A(x) - y\| \geq \|A(x_0) - y\|$, for any $x \in X$ we have

$$\|v_0\| \leq \|v_0 + A_0(x - x_0) + \omega(x_0, x - x_0)\|.$$

In particular, for $x - x_0 = \lambda u_0$ we obtain

$$\begin{aligned} \|v_0\| &\leq \|v_0 + \lambda v_1 + \omega(x_0, \lambda u_0)\| = \\ &= \|v_0 + \lambda v_0 + \lambda(v_1 - v_0) + \omega(x_0, \lambda u_0)\| \leq \\ &\leq |1 + \lambda| \cdot \|v_0\| + |\lambda| \cdot \|v_1 - v_0\| + \|\omega(x_0, \lambda u_0)\|. \end{aligned}$$

Since $\|v_1 - v_0\| < \|v_0\|/2$ and

$$\|\omega(x_0, \lambda u_0)\|/|\lambda| \cdot \|u_0\| \rightarrow 0 \quad \text{as } \lambda \rightarrow 0,$$

we have

$$\|v_0\| < |1 + \lambda| \cdot \|v_0\| + \frac{1}{2}|\lambda| \cdot \|v_0\| + \varepsilon|\lambda| \cdot \|u_0\|,$$

where $\varepsilon \rightarrow 0$ as $\lambda \rightarrow 0$.

The elements v_0 and u_0 do not depend on λ , and by assumption $\|v_0\| > 0$. Put $\|u_0\|/\|v_0\| = C_0$. Then we have

$$\|v_0\| < |1 + \lambda| \cdot \|v_0\| + \frac{1}{2}|\lambda| \cdot \|v_0\| + \varepsilon C_0 |\lambda| \cdot \|v_0\|$$

or ($\|v_0\| > 0$)

$$1 < |1 + \lambda| + \frac{1}{2}|\lambda| + \varepsilon C_0 \cdot |\lambda|, \quad \text{where } \varepsilon \rightarrow 0 \text{ as } \lambda \rightarrow 0.$$

The inequality obtained is contradictory for negative sufficiently small λ . Lemma 2 is proved.

Proof of Theorem 1. Let y be an element of Y . By Lemma 1 there exists in X an element x_0 , a point of minimum of the functional

$$f(x) \equiv \|A(x) - y\|.$$

Let now

$$y - A(x_0) = -v_0 \in (\text{Ker } A_0^*)^\perp.$$

By a known lemma from the theory of bounded linear operators (see (2), p. 516), we have

$$(\text{Ker } A_0^*)^\perp = \overline{RA_0}.$$

Then by Lemma 2 we obtain $y = A(x_0)$.

Theorem 1 can also be formulated as follows:

Theorem 1'. *Let the conditions of Theorem 1 be fulfilled.*

Then, for solvability of equation (2), it is necessary and sufficient that

$$y - A(x_0) \in (\text{Ker}(A'(x_0))^*)^\perp,$$

where x_0 is a point of (absolute) minimum of the functional $f(x) \equiv \|A(x) - y\|$.

From Theorem 1 there follows directly a theorem on the solvability of equation (2).

Theorem 2. *Let Y be a reflexive Banach space and let A , an operator of class C^1 , act from a Banach space X into Y . Let $A(X)$, the range of the operator A , be weakly closed in Y .*

Then, if $\text{Ker}(A'(x))^* = \{0\}$ for every x in X , equation (2) is solvable for every y in Y .

Remark. This theorem (in the case of a reflexive Banach space Y) generalizes the well-known theorem on the solvability of a linear equation (1). In the case of a linear operator L , the weakly closed range of the operator L coincides with the closed (strong) range of the operator L .

The “orthogonality” condition $y - A(x) \in (\text{Ker}(A'(x))^*)^\perp$ has a form different from the corresponding condition for the case of a linear operator.

Let us define a class of operators $A(x)$ for which the “orthogonality” condition takes the form $y \in (\text{Ker}(A'(x))^*)^\perp$, i.e., for which $A(x) \in (\text{Ker}(A'(x))^*)^\perp$.

Theorem 3. In order that $A(x) \in (\text{Ker}(A'(x))^*)^\perp$, it is sufficient and, if the linear operator $A'(x)$ (for fixed x) has a closed range, necessary that there exist a mapping $\Phi(x)$ from X into X such that the operator $A(x)$ of class C^1 has the form

$$A(x) = A'(x) \cdot \Phi(x). \quad (3)$$

Remark. The last relation for an operator $A(x)$ of class C^1 is equivalent to the following:

$$A(x) = \left. \frac{d}{dt} A(x + t\Phi(x)) \right|_{t=0}, \quad (4)$$

where t is a real number.

Proof. Sufficiency. We have

$$\langle y^*, A(x) \rangle = \langle y^*, A'(x) \cdot \Phi(x) \rangle = \langle (A'(x))^* y^*, \Phi(x) \rangle.$$

Therefore, if $y^* \in \text{Ker}(A'(x))^*$ (x fixed), we obtain $\langle y^*, A(x) \rangle = 0$, i.e. $A(x) \in (\text{Ker}(A'(x))^*)^\perp$.

Necessity. Let $A(x) \in (\text{Ker}(A'(x))^*)^\perp$. Since the range of the linear operator $A'(x)$ (x fixed) is closed and the element $A(x)$ belongs to $(\text{Ker}(A'(x))^*)^\perp$, by Banach's theorem (see ⁽³⁾, p. 284) the equation $A'(x) \cdot h = A(x)$ is solvable with respect to h . Denoting the solution h , depending on x , by $\Phi(x)$, we obtain the second assertion of the theorem.

Remark. It is clear from the proof that the mapping $\Phi(x)$ is, generally speaking, not uniquely determined.

Let us note that a positively homogeneous operator $A(x)$, $A(kx) = k^a A(x)$ ($k > 0$, $a > 0$), of class C^1 satisfies the condition of Theorem 3. Indeed, it is enough in relation (4) to put $\Phi(x) = x/a$.

From Theorems 1 and 3 we obtain the following theorem.

Theorem 4. Suppose that the conditions of Theorem 1 are fulfilled and the operator $A(x)$ admits the representation (3).

Then, for the solvability of equation (2) it is necessary and sufficient that $y \in (\text{Ker}(A'(x_0))^*)^\perp$, where x_0 is a point of (absolute) minimum of the functional $f(x) = \|A(x) - y\|$.

In those cases when the range of the operator A does not coincide with the space Y , the operator A can be extended by additionally introducing a Banach space Z —a space of “potentials.” In the theory of boundary-value problems such an extension corresponds to the introduction of additional potentials (see ⁽⁴⁾).

Let A be an operator of class C^1 acting from a Banach space X into a Banach space Y , and let B be an operator of class C^1 acting from a Banach space Z into Y , with $B(0) = 0$. Consider the Banach space $X \times Z$, for example, with norm $\|(x, z)\| = \|x\| + \|z\|$, and define the extension $\mathfrak{A} : X \times Z \rightarrow Y$ of the operator A by the formula $\mathfrak{A}(x, z) = \mathfrak{A}(x, 0) + \mathfrak{A}(0, z)$, where $\mathfrak{A}(x, 0) = A(x)$, $\mathfrak{A}(0, z) = B(z)$. For the derivative $\mathfrak{A}'(x, z)$ we have $\mathfrak{A}'(x, z) \cdot (u, w) = A'(x) \cdot u + B'(z) \cdot w$. The linear operator

$(\mathfrak{A}'(x, z))^*$ (for fixed x, z) has kernel $\text{Ker}(\mathfrak{A}'(x, z))^*$, determined by the relation

$$\text{Ker}(\mathfrak{A}'(x, z))^* = \text{Ker}(A'(x))^* \cap \text{Ker}(B'(z))^*.$$

Applying Theorem 2, we obtain

Theorem 5. Let Y be a reflexive Banach space. Let $\mathfrak{A} : X \times Z \rightarrow Y$ be an operator of class C^1 , defined by the relation $\mathfrak{A}(x, z) = A(x) + B(z)$, and suppose that it has in Y a weakly closed range.

Then, if $\text{Ker}(A'(x))^* \cap \text{Ker}(B'(z))^* = \{0\}$ for all (x, z) in $X \times Z$, the equation $A(x) + B(z) = y$ is solvable for every y in Y .

A sufficient condition for the weak closedness of the range of the operator A is contained in the following lemma.

Lemma 3. Let X be a reflexive Banach space. Let the operator $A : X \rightarrow Y$ be defined on all of X and be weakly closed (i.e., from $x_n \rightarrow x_0$ weakly in X and $A(x_n) \rightarrow y_0$ weakly in Y as $n \rightarrow \infty$ it follows that $y_0 = A(x_0)$), and such that for every bounded set $G \subseteq A(X)$ there exists a bounded set $\Omega \subset X$ such that $A(\Omega) = G$.

Then $A(X)$, the range of the operator A , is weakly closed in Y .

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Moscow
Power Engineering Institute

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Note: Figure translations are in progress. See original paper for figures.

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