

EMBEDDING COMPACTA IN EUCLIDEAN SPACE

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Abstract

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MATHEMATICS

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EMBEDDING COMPACTA IN EUCLIDEAN SPACE

(Presented by Academician P. S. Aleksandrov on 13 XII 1968)

1. An embedding of a space X in a space Y is an arbitrary homeomorphic mapping $X \rightarrow Y$; we shall omit the notation of the mapping and by an embedding understand simply the inclusion $X \subset Y$.

Here we shall obtain some results for embeddings of compacta of dimension $\leq n - 3$ in Euclidean space E^n . The basic concept is the embedding dimension of a compactum K in the space E^n , $K \subset E^n$, denoted by $\text{dem } K$, (dem = dimension of embedding), which is an integer-valued invariant, unchanged under homeomorphisms of E^n onto itself (nonnegative for nonempty compacta), reminiscent of the characterization of the ordinary topological dimension $\dim K$ of a compactum K by P. S. Aleksandrov's method of approximating compacta by polyhedra (see ⁽¹⁾). In the case $\dim K \leq n - 3$ this invariant depends on the embedding of K in E^n .

We shall call an embedding $K \subset E^n$, where $\dim K \leq n - 3$, tame if $\text{dem } K = \dim K$, and wild otherwise. (We note that always $\text{dem } K \geq \dim K$.)

In the case of zero-dimensional compacta and r -dimensional topological polyhedra in E^n ($2r + 2 \leq n$), the definition of a tame embedding introduced by us coincides with the previously known one (see ⁽²⁾). For nontrivial dimensions it is unknown whether the concept of a tame embedding introduced by us coincides with the classical one (see unsolved questions III, III'). Menger's problem on universal compacta is also considered (see ⁽³⁾), which turns out to be connected with embeddings.

Let us recall some simplest notation and concepts. $U(K, \varepsilon)$ denotes the ε -neighborhood of the compactum K in E^n . A compactum K lying in E^n is called cellularly separated if in every neighborhood U of the compactum K one can inscribe such a neighborhood V that $U \supset \bar{V} \supset V \supset K$, and $\bar{V}$ is homeomorphic to a disjoint union of a finite number of topological cubes. A homotopy $F_t : X \rightarrow X$, $t \in [0, 1]$, is called a pseudoisotopy if for every $t < 1$, F_t is a homeomorphism of the space X onto itself, and an isotopy if F_t is a homeomorphism for every t .

Definition 1. The **embedding dimension of a nonempty compactum** $K \subset E^n$ is the least integer $\text{dem } K = r \geq 0$ such that for every $\varepsilon > 0$ there exists an ε -pseudoisotopy $F_t : E^n \rightarrow E^n$, $t \in [0, 1]$, of the space E^n onto itself, fixed outside $U(K, \varepsilon)$, and such that $F_1(K) = P$, where P is a rectilinear polyhedron in E^n , $\dim P = r$.

The embedding dimension of the empty compactum in E^n is by definition equal to -1 .

Main theorem. If $K \subset E^n$, $n \neq 3$, $\dim K \geq n - 2$, then always $\text{dem } K = \dim K$; if $n \geq 5$ and $\dim K \leq n - 3$, then the embedding dimension assumes two (and only two) values: either $\dim K$, or $n - 2$, the latter occurring if and only if $E^n \setminus K$ does not have the property 1-*ULC*.

2. Definition 2. The **dual embedding dimension of a nonempty compactum** $K \subset E^n$ is called the least

an integer $\text{Dem } K = r \geq 0$ such that, whatever rectilinear polyhedron $P \subset E^n$, $\dim P \leq n - r - 1$, for every $\varepsilon > 0$ there exists an ε -isotopy $F_t : E^n \rightarrow E^n$, $t \in [0, 1]$, fixed outside $U(K \cap P, \varepsilon)$ and such that $F_1(K) \cap P = \Lambda$. The dual embedding dimension of the empty compactum in E^n is, by definition, equal to -1 .

Let M^n be the geometric cube in E^n . Divide the cube M^n into 3^n equal cubes of first rank by hyperplanes drawn perpendicular to the edges of the cube M^n at the points dividing the edges into three equal parts; from these 3^n cubes select those which intersect the r -dimensional skeleton of the cube M^n , and denote their union by $M_{r,1}^n$. Each cube entering into $M_{r,1}^n$ is divided in an analogous manner into 3^n cubes of second rank; the sum of all the analogously selected cubes of second rank is denoted by $M_{r,2}^n$, and so on. We obtain a decreasing sequence of closed sets

$$M^n \supset M_{r,1}^n \supset M_{r,2}^n \supset \dots$$

The compactum $M_r^n = \bigcap_{k=1}^{\infty} M_{r,k}^n$ is the Menger compactum (see (3)).

For a compactum $K \subset E^n$ the following assertions follow from the definitions or were known earlier.

- (a) Each of the conditions $\text{dem } K = 0$ or $\text{Dem } K = 0$, respectively, is equivalent to the conjunction of two conditions: (1⁰) $\dim K = 0$; (2⁰) K is cellularly decomposable.
- b) If $n \geq 3$ and A_0 is a wild zero-dimensional compactum in E^n , $A_0 \subset E^n$, then $\text{dem } A_0 = \text{Dem } A_0 = n - 2$.
- (c) There exists a compactum K in E^3 such that $\dim K = 1$, whereas $\text{dem } K = \text{Dem } K = 2$, for example, the compactum constructed by Bothe (4).

(d) The embedding dimensions $\text{dem } K$, $\text{Dem } K$ possess the properties of invariance and monotonicity, i.e. the corresponding value of the embedding dimension does not change if the space is subjected to a homeomorphism onto itself, and, on passing to a subcompactum, the corresponding value of the embedding dimension does not increase.

(e) $\text{dem}(M_r^n) = \text{Dem}(M_r^n) = \dim(M_r^n) = r$, $M_r^n \subset E^n$.

3. The **main theorem** can be formulated as follows:

If a compactum K lies in E^n , $n \neq 3$, $\dim K \geq n-2$, then always $\text{dem } K = \dim K$; if $n \geq 5$ and $\dim K \leq n-3$, then the compactum K is tame in E^n if and only if $E^n \setminus K$ has the property 1-*ULC*; but if $E^n \setminus K$ does not have the property 1-*ULC*, then always $\text{dem } K = n-2$.

Remark. By virtue of Bing's theorem this assertion is also true for $n = 3$ and $\dim K = 0$ (5). An analogous assertion was earlier proved by McMillan for zero-dimensional compacta in E^n , $n \geq 5$ (6).

Theorem 1. For every compactum K lying in E^n , $\text{dem } K = \text{Dem } K$.

Theorem 2. Let compacta K and C lie in the space E^n , $n \geq 5$, $\dim K = r$, $2r+2 \leq n$, and let $h : K \rightarrow C$ be a homeomorphism; if $E^n \setminus K$ and $E^n \setminus C$ have the property 1-*ULC*, then for every $\varepsilon > \rho(h, 1)$ there exists an ε -isotopy $F_t : E^n \rightarrow E^n$, $t \in [0, 1]$, fixed outside the prescribed neighborhood $V \supset X(K, h, C)$ and such that $F_1|_K = h$. (Here $X(K, h, C)$ denotes the union of the segments $xh(x)$, $x \in K$.)

Let us note a particular case of Theorem 2, obtained earlier by Bryant and Seebeck: let a topological polyhedron P lie in E^n , $n \geq 5$, $\dim P = r$, $2r+2 \leq n$; if $E^n \setminus P$ has the property 1-*ULC*, then for every $\varepsilon > 0$ there exists an ε -isotopy of the space E^n onto itself, fixed outside the ε -neighborhood of the polyhedron P and carrying it into a piecewise-linear one (7).

Theorem 3. If a compactum K lies in E^n , then the condition $\text{Dem } K = r$ is equivalent to the fact that $M_r^n \subset E^n$ is a Menger compactum of the least dimension for which there exists an isotopy $F_t : E^n \rightarrow E^n$, $t \in [0, 1]$, fixed outside some bounded domain V and such that $F_1(K) \subset M_r^n$.

A somewhat weaker assertion was proved earlier by Bothe (8).

Corollary 1. The Menger compactum M_r^n , $M_r^n \subset E^n$, $n \geq 5$, $r \leq n-3$, is universal for all r -dimensional compacta in E^n , $K \subset E^n$, such that $E^n \setminus K$ has the property 1-*ULC*; moreover, in this case there exists an isotopy $F_t : E^n \rightarrow E^n$, $t \in [0, 1]$, fixed outside some bounded domain V and such that $F_1(K) \subset M_r^n$.

Corollary 2. The Menger compactum M_r^n , $M_r^n \subset E^n$, $n \geq 5$, $r \leq n-3$, is universal for all r -dimensional compacta K lying in the hyperplane E^{n-1} , $K \subset E^{n-1} \subset E^n$; moreover, in this case there exists an isotopy $F_t : E^n \rightarrow E^n$, $t \in [0, 1]$, fixed outside some bounded domain V and such that $F_1(K) \subset M_r^n$.

4. For the proof of the main theorem the absorption lemma ⁽⁹⁾ is used. We briefly describe the idea of the proof.

We shall prove that if $\dim K \leq n-3$ and $E^n \setminus K$ has the property 1-*ULC*, then $\text{Dem } K \leq \dim K^*$; hence, taking into account that always $\text{Dem } K \geq \dim K$, we shall have $\text{Dem } K = \dim K$. All the remaining assertions of the theorem follow easily from the definitions. The proof of the indicated assertion is carried out by induction on the increasing dimension of the polyhedron P , $\dim P = p$, from which, according to Definition 2, the compactum K is removed, $0 \leq p \leq n-r-1$, where $r = \dim K$, $n \geq 5$. Since $\dim K \leq n-3$, the inductive assertion is fulfilled for the values $p = 0, 1$. Assuming that it is fulfilled for some p , $0 \leq p < n-r-1$, we prove that it is fulfilled for $p+1$.

Let P be a rectilinear polyhedron in E^n , $\dim P = p+1$. By the induction hypothesis we may assume that $K \cap P^p = \Lambda$, where P^p is the p -skeleton of the polyhedron P . Let Δ^{p+1} be some simplex in the given ε -triangulation of the polyhedron P such that $\Delta^{p+1} \cap K \neq \Lambda$; since $\partial(\Delta^{p+1}) \cap K = \Lambda$, there exists a simplex Δ_0^{p+1} , concentric with the simplex Δ^{p+1} , $\text{int } \Delta^{p+1} \supset \Delta_0^{p+1}$, and, if $C = \Delta^{p+1} \setminus \text{int } \Delta_0^{p+1}$, then $C \cap K = \Lambda$.

Let $L^n(\Delta^{p+1}) = L^n$ be a convex polyhedral n -cube in E^n such that $L^n \cap P = \Delta^{p+1}$, $\partial \Delta^{p+1} \subset \partial L^n$, $\text{int } \Delta^{p+1} \subset \text{int } L^n$. Since $E^n \setminus K$ has the property 1-*ULC* and $p \leq n-r-2$, one can show that the sphere $\partial \Delta_0^{p+1}$ contracts to a point in the set $(\text{int } L^n) \setminus K$; therefore there exists a polyhedral homotopy film Π_0 in $(\text{int } L^n) \setminus K$, effecting the contractibility of the p -dimensional sphere $\partial \Delta_0^{p+1}$ in $(\text{int } L^n) \setminus K$ to a point; putting it in general position, we may assume that the self-intersection of this film has minimal dimension. If the self-intersection is empty and $\Pi_0 \cap C = \partial \Pi_0 = \partial \Delta_0^{p+1}$, then the resulting piecewise-linear disk $\Pi = \Pi_0 \cup C$ forms a piecewise-linear pair of cells (L^n, Π) ; if it is unknotted, then Π is carried onto Δ^{p+1} fixed outside $\text{int } L^n$, whereby the compactum K is removed from Δ^{p+1} . If, however, the self-intersection of Π_0 is nonempty, then either the absorption lemma is applied in a special way, or Ziman's device (with cones) is applied directly to eliminate the self-intersection.

5. Questions. (I). Let a compactum K lie in E^n and $\dim K \leq n-3$. Does there exist a homeomorphism $h : K \rightarrow E^n$ such that $E^n \setminus h(K)$ has the property 1-*ULC*?

(I'). Let a compactum K lie in E^n and $\dim K \leq n-3$. Does there exist, for every $\varepsilon > 0$, a homeomorphic ε -shift $h_\varepsilon : K \rightarrow E^n$ such that $E^n \setminus h(K)$ has the property 1-*ULC*?

(II). Let P be a topological polyhedron in E^n , $\dim P \leq n-3$. Does there exist a piecewise-linear embedding $h : P \rightarrow E^n$?

(II'). Let P be a topological polyhedron in E^n , $\dim P \leq n-3$. Does there exist, for every $\varepsilon > 0$, a homeomorphic piecewise-linear ε -shift $h_\varepsilon : P \rightarrow E^n$?

* According to Theorem 1, in the formulation of the main theorem we may

replace $\text{dem } K$ by $\text{Dem } K$.

(III). Let P be a topological polyhedron in E^n , $\dim P \leq n-3$, and suppose that $E^n \setminus P$ has the property 1-*ULC*. Will the polyhedron P be tame in E^n ?

(III'). Let P be a topological polyhedron in E^n , $\dim P \leq n-3$, and suppose that $E^n \setminus P$ has the property 1-*ULC*. Can the polyhedron P be carried into a piecewise-linear one by means of an ε -isotopy of the space E^n , fixed outside the ε -neighborhood of P ?

(We note that questions I–III' have positive solutions in trivial dimensions.)

(IV). Let a compactum K lie in E^n , $K \subset E^n$, $n = 4$. Will the equality $\dim K = \text{Dem } K$ hold when $\dim K = 0$ or 1 and $E^n \setminus K$ has the property 1-*ULC*?

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Note: Figure translations are in progress. See original paper for figures.

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