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Abstract

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MATHEMATICAL PHYSICS

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ASYMPTOTIC EXPANSIONS IN THE THEORY OF DIFFRACTION OF A PLANE WAVE BY A STRIP

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1. In this paper, on the basis of the method of edge waves ⁽¹⁾ and the factorization method ⁽²⁾, an asymptotic solution is obtained for the problem of diffraction by a strip, with accuracy up to any prescribed degree $(kl)^{-n/2}$, for $2kl \gg n$, where k is the wave number in free space, $2l$ is the width of the strip, and n is any positive integer. The case considered is that in which a plane electromagnetic wave

$$H_x = \exp \left[-ik \left(za_0 + y\sqrt{1-a_0^2} \right) - i\omega t \right], \quad E_x = 0, \quad a_0 = \cos \vartheta_0, \quad (1)$$

$$0 \leq \vartheta_0 \leq \pi,$$

excites a perfectly conducting strip, whose orientation in free space is determined by the relations $y = 0$, $-l \leq z \leq l$, $-\infty \leq x \leq \infty$. In other words, the Neumann boundary-value problem for the Helmholtz equation is investigated. The paper computes the surface-current density on the strip and the scattering characteristic in the far zone. The method presented is readily generalized to the case of the other polarization for the strip and can be applied to a number of other problems.

2. Let us represent the surface-current density in the form

$$j_z = \frac{c}{2\pi} J(z, a_0), \quad J(z, a_0) = -e^{-ikza_0} + \tilde{J}(z+l, a_0) + \tilde{J}(l-z, -a_0), \quad (2)$$

where

$$\tilde{J}(z, a_0) = \sum_1^{\infty} j_n(z, a_0) \quad (3)$$

describes the edge waves arising under successive diffraction

at the edges of the strip. Put $j_n(z, a_0) = \int_{-\infty}^{\infty} e^{izt} F_n(t, ka_0) dt$, and denote by

$E_n(z, y, a_0)$ the electric vector of the field radiated by this current. Then the conditions $J(z, a_0) = 0$ for $|z| > l$ and $E_{nz}(z, 0, a_0) = 0$ for $z > 0$ lead to a system of functional equations with respect to $F_n(t, ka_0)$ (see, for example, ⁽³⁾). The solution of these equations is found by the factorization method. It is equivalent to Schwarzschild's solution ⁽⁴⁾ and can also be obtained by the method of successive approximations from the integral equation of the second kind formulated in ⁽⁵⁾.

3. Having determined the functions F_n , we find

$$\tilde{J}(z, a_0) = -e^{ikz} \sum_{n=1}^{\infty} (-1)^n \chi_n(z, -\beta_n) e^{iq(n-1) - ix\beta_n}, \quad \beta_n = (-1)^n a_0, \quad (4)$$

where

$$\begin{aligned} \chi_1(z, \alpha) e^{ikz} &= -\frac{\sqrt{1+\alpha}}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{ikzt}}{\sqrt{1-t}} \frac{dt}{t+\alpha}; \\ \chi_2(z, \alpha) e^{iq+ikz} &= -\frac{\sqrt{1+\alpha}}{2(\pi i)^2} \int_1^{1+i\infty} \frac{h(t_1) dt_1}{t_1 + \alpha} \int_{-\infty}^{\infty} \frac{e^{ikzt_2}}{\sqrt{1-t_2}} \frac{dt_2}{t_2 + t_1}, \quad (5) \\ \chi_n(z, \alpha) e^{iq(n-1)+ikz} &= -\frac{\sqrt{1+\alpha}}{2(\pi i)^n} \times \int_1^{1+i\infty} \dots \int_1^{1+i\infty} \frac{h(t_1)h(t_2) \dots h(t_{n-1}) dt_1 dt_2 \dots dt_{n-1}}{(t_1 + \alpha)(t_2 + t_1)(t_3 + t_2) \dots (t_{n-1} + t_{n-2})} \int_{-\infty}^{\infty} \frac{e^{ikzt_n}}{\sqrt{1-t_n}} \frac{dt_n}{t_n + t_{n-1}}; \\ h(t) &= \sqrt{\frac{1+t}{1-t}} e^{iqt}, \quad q = 2\chi, \quad \chi = kl. \quad (7) \end{aligned}$$

The integrals with limits from 1 to $1+i\infty$ are taken along the left bank of the cut $1 \rightarrow 1+i\infty$. In addition, the contour of integration passes above the poles $t = -\alpha$, $t_2 = -t_1$, $t_n = -t_{n-1}$. The contour passing along the real axis goes around the branch points: the point $t = -1$ from above, and the point $t = 1$ from below.

Knowing the current, one can determine the scattered field. In the far zone it is equal to

$$H_x = \Phi(\alpha, \alpha_0) \frac{e^{i(kr+\pi/4)}}{\sqrt{2\pi kr}}, \quad \alpha = \cos \vartheta, \quad r = \sqrt{y^2 + z^2}, \quad y \geq 0, \quad (8)$$

where

$$\Phi(\alpha, \alpha_0) = \tilde{\Phi}(\alpha, \alpha_0) + \tilde{\Phi}(-\alpha, -\alpha_0), \quad (9)$$

$$\tilde{\Phi}(\alpha, \alpha_0) = e^{i\chi\alpha} \left\{ \sum_{n=1}^{\infty} (-1)^n \varphi_n(\alpha, -\beta_n) e^{iq(n-1)-i\chi\beta_n} \right\}, \quad \beta_n = (-1)^n \alpha_0; \quad (10)$$

$$\varphi_1(\alpha, \alpha_0) = \frac{\sqrt{1+\alpha}\sqrt{1+\alpha_0}}{\alpha + \alpha_0},$$

$$\varphi_n(\alpha, \alpha_0) = -ik\sqrt{1-\alpha^2} \int_{-\infty}^{\infty} e^{ik(1-\alpha)z} \chi_n(z, \alpha_0) dz. \quad (11)$$

The series (4), (10) converge absolutely and uniformly for any values $kl > 0$ and constitute a rigorous solution of the boundary-value problem, expressed in terms of new special functions. Unlike the classical series in Mathieu functions, they converge the faster, the larger the parameter kl .

4. Generalizing the well-known Watson lemma [6] to multiple integrals, one can obtain the following asymptotic expansions as $q \rightarrow \infty$:

$$\chi_2(z, \alpha) \sim \varphi(q, \alpha) \varphi(kz, 1) + \sum_{m=1}^{\infty} \frac{\varphi_q^{(m)}(q, \alpha)}{\pi i^{m+1}} \int_1^{1+i\infty} \frac{e^{ikz(t-1)}}{\sqrt{1-t}} c_m(t) dt, \quad z \geq 0; \quad (12)$$

$$\varphi_2(\alpha, \alpha_0) \sim \frac{1}{\alpha - \alpha_0} \sum_{m=0}^{\infty} \frac{a_m}{i^m} [\sqrt{1+\alpha} \varphi_q^{(m)}(q, \alpha_0) - \sqrt{1+\alpha_0} \varphi_q^{(m)}(q, \alpha)]; \quad (13)$$

$$\chi_n(z, \alpha) \sim \varphi(q, \alpha) \varphi(kz, 1) \varphi^{n-2}(q, 1) + \frac{\sqrt{1+\alpha}}{(\pi i)^n} \sum_{m=1}^{\infty} \frac{1}{m!} \int_1^{1+i\infty} \frac{e^{ikz(t_n-1)}}{\sqrt{1-t_n}} dt_n \int_1^{1+i\infty} \dots \int_1^{1+i\infty} \exp \left[iq \sum_{s=1}^{n-1} (t_s - 1) \right]$$

$$u_m(t_1, t_2, \dots, t_n) = \left[\sum_{s=1}^{n-1} (t_s - 1) \frac{\partial}{\partial t_s} \right]^m f_n(1, 1, \dots, 1, t_n);$$

$$f_n(t_1, t_2, \dots, t_n) = \frac{\sqrt{1+t_1} \prod_{s=2}^{n-1} (1+t_s)^{3/2}}{(t_1+t_2)(t_2+t_3) \dots (t_{n-1}+t_n)}, \quad n = 3, 4, \dots, \quad z \geq 0;$$

$$\varphi_n(\alpha, \alpha_0) \sim \varphi(q, \alpha) \varphi(q, \alpha_0) \varphi^{n-3}(q, 1) + \frac{\sqrt{1+\alpha} \sqrt{1+\alpha_0}}{(i\pi)^{n-1}} \sum_{m=1}^{\infty} \frac{1}{m!} \int_1^{1+i\infty} \dots \int_1^{1+i\infty} \exp \left[iq \sum_{s=1}^{n-1} (t_s - 1) \right] \times \times \frac{1}{(t_1}$$

$$\hat{u}_m(t_1, t_2, \dots, t_{n-1}) = \left[\sum_{s=1}^{n-1} (t_s - 1) \frac{\partial}{\partial t_s} \right]^m \hat{f}_n(1, 1, \dots, 1),$$

$$\hat{f}_n(t_1, t_2, \dots, t_{n-1}) = \frac{\sqrt{1+t_1} \sqrt{1+t_{n-1}} \prod_{s=2}^{n-2} (1+t_s)^{3/2}}{(t_1+t_2)(t_2+t_3) \dots (t_{n-2}+t_{n-1})},$$

$$n = 4, 5, \dots, \quad \hat{f}_3(t_1, t_2) = \sqrt{1+t_1} \sqrt{1+t_2} (t_1+t_2)^{-1},$$

where $c_m(t)$ and a_m are the coefficients of the Taylor series

$$(1+x)^{1/2}(x+t)^{-1} = \sum_{m=0}^{\infty} c_m(t)(x-1)^m, \quad \sqrt{1+t} = \sum_{m=0}^{\infty} a_m(t-1)^m, \quad (16)$$

and the function $\varphi(q, \alpha)$ is the Fresnel integral

$$\varphi(q, \alpha) = \frac{2}{\sqrt{\pi}} e^{-i\pi/4} e^{-iq(1+\alpha)} \int_{\sqrt{q(1+\alpha)}}^{\infty} e^{it^2} dt. \quad (17)$$

The series (12)–(15) are expressed in terms of single integrals of the form

$$\frac{\partial^m \varphi(q, \alpha)}{\partial q^m} = i^m \frac{\sqrt{1+\alpha}}{i\pi} \int_1^{1+i\infty} \frac{e^{iq(t-1)}}{\sqrt{1-t}} \cdot \frac{(t-1)^m}{t+\alpha} dt,$$

$$\varphi_m(z) = \frac{\sqrt{2}}{i\pi} \int_1^{1+i\infty} \frac{e^{iq(t-1)}}{\sqrt{1-t}} \frac{dt}{(t+1)^m}, \quad (18)$$

where

$$\varphi_1(z) = \varphi(z, 1), \quad \varphi_m(z) = i^{m-1} e^{-2iz} \int_{\infty}^z ds_1 \int_{\infty}^{s_1} ds_2 \dots \int_{\infty}^{s_{m-2}} ds_{m-2} \times \times \int_{\infty}^{s_{m-2}} e^{2is_{m-1}} \varphi(s_{m-1}, 1) ds_{m-1},$$

5. With the aid of these asymptotic expansions, formulas for the current and the scattering pattern have been obtained in the n -th approximation:

$$J(z, \alpha_0) = \tilde{J}_n(z, \alpha_0) + \tilde{R}_n(z, \alpha_0), \quad \tilde{\Phi}(\alpha, \alpha_0) = \tilde{\Phi}_n(\alpha, \alpha_0) + \tilde{P}_n(\alpha, \alpha_0), \quad (19)$$

where $\tilde{R}_n(z, \alpha_0)$ and $\tilde{P}_n(\alpha, \alpha_0)$ characterize the absolute error of the quantities $\tilde{J}_n(z, \alpha_0)$ and $\tilde{\Phi}_n(\alpha, \alpha_0)$. In the first approximation,

$$\tilde{J}_1(z, \alpha_0) = \varphi(kz, \alpha_0) e^{ix\alpha_0 + ikz} - \frac{e^{iq}}{D_1} [\varphi(q, -\alpha_0) e^{-ix\alpha_0} - \varphi(q, \alpha_0) \varphi(q, 1) e^{ix\alpha_0 + iq}] \varphi(kz, 1) e^{ikz}; \quad (20)$$

$$\tilde{\Phi}_1(\alpha, \alpha_0) = -\frac{\sqrt{1+\alpha}\sqrt{1+\alpha_0}}{\alpha+\alpha_0} e^{ix(\alpha+\alpha_0)} + \frac{e^{iq}}{\alpha+\alpha_0} \left[(1+\alpha_0) \sqrt{\frac{1+\alpha}{2}} \varphi(q, -\alpha_0) - (1-\alpha) \sqrt{\frac{1-\alpha_0}{2}} \varphi(q, \alpha) \right] e^{ix\alpha}$$

where

$$D_1 = 1 - \varphi^2(q, 1) e^{2iq}. \quad (22)$$

The absolute error of these expressions for $-1 \leq a, a_0 \leq 1$ is determined by the formulas

$$\tilde{R}_1(z, a_0) = O \left[\frac{1 + \sqrt{kz}}{1 + kz} \frac{q^{-1/2}}{1 + q(1 - a_0)} \right] \quad \text{for } q \gg 1 \text{ and } 0 \leq z \leq \infty, \quad (23)$$

$$\tilde{P}_1(a, a_0) = \frac{O(\sqrt{q})}{[1 - q(1 + a)][1 + q(1 - a_0)]} \quad \text{for } q \gg 1. \quad (24)$$

In the n -th approximation ($n = 2, 3, \dots$) we have

$$\tilde{J}_n(z, a_0) = -e^{ikz} \sum_{m=1}^n (-1)^m \chi_m(z, -\beta_m) e^{iq(m-1) - ix\beta_m} + (-1)^n \frac{e^{iqn}}{\widehat{D}_n} \left[\chi_n(2l, \beta_n) e^{ix\beta_n} - \chi_n(2l, -\beta_n) \frac{\chi_n(2l, 1)}{\chi_{n-1}(2l, 1)} e^{ix\beta_n} \right]$$

$$\tilde{\Phi}_n(a, a_0) = e^{ixa} \sum_{m=1}^n (-1)^m \varphi_m(a, -\beta_m) e^{iq(m-1) - ix\beta_m} - (-1)^n \frac{e^{iqn}}{D_n} \left[\varphi_n(1, \beta_n) e^{ix\beta_n} - \varphi_n(1, -\beta_n) \frac{\varphi_n(1, 1)}{\varphi_{n-1}(1, 1)} e^{ix\beta_n} \right] \quad (26)$$

where

$$\beta_n = (-1)^n a_0, \quad \widehat{D}_n = 1 - \left[\frac{\chi_n(2l, 1)}{\chi_{n-1}(2l, 1)} \right]^2 e^{2iq}, \quad D_n = 1 - \left[\frac{\varphi_n(1, 1)}{\varphi_{n-1}(1, 1)} \right]^2 e^{2iq}, \quad (27)$$

moreover $|\widehat{D}_n - D_n| \leq O(q^{-2})$ for $q \gg n$. The functions

$$J_n(z, a_0) = -e^{-ikza_0} + \widetilde{J}_n(z + l, a_0) + \widetilde{J}_n(l - z, -a_0),$$

$$\Phi_n(a, a_0) = \widetilde{\Phi}_n(a, a_0) + \widetilde{\Phi}_n(-a, -a_0) \quad (28)$$

satisfy the boundary conditions

$$J_n(\pm l, a_0) = J_n(z, \pm 1) = 0,$$

$$\Phi_n(\pm 1, a_0) = \Phi_n(a, \pm 1) = 0, \quad n = 1, 2, \dots, \quad (29)$$

and the reciprocity principle

$$\Phi_n(a, a_0) = \Phi_n(a_0, a). \quad (30)$$

The absolute error of $\widetilde{J}_n$ and $\widetilde{\Phi}_n$ has the following upper estimate:

$$|\widetilde{R}_n(z, a_0)| \leq O \left\{ \frac{1 + \sqrt{kz}}{1 + kz} \frac{q^{-(n+1)/2}}{1 + \sqrt{q}[1 + (-1)^n a_0]} \right\},$$

$$|\widetilde{P}_n(a, a_0)| \leq \frac{O(q^{-n/2})}{[1 + \sqrt{q}(1 + a)][1 + \sqrt{q}(1 + (-1)^n a_0)]} \quad (31)$$

for $0 \leq z \leq \infty$ and $q \gg n$, $n = 2, 3, \dots$

Let us note that formula (20) for the current was obtained by us earlier by the method of the parabolic equation ⁽³⁾. It is also interesting that the formula of the first approximation for the scattering pattern (21) is just as simple as the known expressions from ^(7, 8), but has a higher degree of accuracy.

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