

ON THE SPECTRUM OF SINGULAR INTEGRALS ON DOMAINS WITH BOUNDARY

R. Denchev

1969

SovietRxiv

View the original and related papers at <https://sovietsrxiv.org/items/ru-196901.24889>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.944

MATHEMATICS

R. DENCHEV

ON THE SPECTRUM OF SINGULAR INTEGRALS ON DOMAINS WITH BOUNDARY

(Presented by Academician S. L. Sobolev on 22 X 1968)

Let Ω be a domain of m -dimensional Euclidean space, bounded by a finite number of simple closed surfaces of Lyapunov type, not intersecting one another. Introduce the operator of extension by zero outside Ω

$$P_{\Omega} : L_2(\Omega) \ni f(x) \rightarrow (P_{\Omega}f)(x) = \begin{cases} f(x) & \text{for } x \in \Omega, \\ 0 & \text{for } x \notin \Omega \end{cases} \in L_2(E_m)$$

and the operator of restriction to Ω

$$R_{\Omega} : L_2(E_m) \ni f(x) \rightarrow (R_{\Omega}f)(x) = \{f(x) \text{ for } x \in \Omega\} \in L_2(\Omega).$$

Let $\mathcal{A}$ be a singular integral operator with symbol $\sigma(x, \xi)$. The function $\sigma(x, \xi)$ is defined for all x and $\xi \in E_m$, $\xi \neq 0$, is positively homogeneous in ξ of degree zero, and on the unit sphere $\Sigma = \{\xi : |\xi| = 1\}$ belongs to the space $H_2(\Sigma)$. Suppose that, with respect to x , the function $\sigma(x, \xi)$ is sufficiently smooth. As is known, the operator $\mathcal{A}$ maps $L_2(E_m)$ into itself.

The present work is devoted to the study of the essential spectrum of the operator

$$A = R_{\Omega}\mathcal{A}P_{\Omega}.$$

Obviously, A maps $L_2(\Omega)$ into itself. A point λ is called a point of the essential spectrum of the operator A if the operator $A - \lambda I$ is not Noetherian.

We shall use the necessary and sufficient condition for the Noetherian property of singular integral operators contained in ⁽¹⁾.

Let $m \geq 2$, and let x_0 be a point of the boundary of Ω . Draw at x_0 the unit normals: the inward $n_{x_0}^i$ and the outward $n_{x_0}^e$. Move $n_{x_0}^i$ and $n_{x_0}^e$ to the origin of coordinates. Their endpoints mark on the unit sphere Σ two points. Connect these points by all possible semicircles l_{x_0} . Introduce the quantity

$$d_{x_0}^{l_{x_0}}(\lambda) = \{\arg[\sigma(x, \xi) - \lambda]\}_{l_{x_0}}.$$

The braces denote the change of the quantity in the braces when ξ varies along the semicircle l_{x_0} .

In the case $m > 2$ all paths l_{x_0} are homotopic, and $d_{x_0}^{l_{x_0}}(\lambda)$ does not depend on l_{x_0} . We denote the common value $d_{x_0}^{l_{x_0}}(\lambda)$ by $d_{x_0}(\lambda)$.

In the case $m = 2$ there are two classes of nonhomotopic paths, and we obtain two numbers $d_{x_0}^+(\lambda)$ and $d_{x_0}^-(\lambda)$. In the case $m = 1$ we denote $d_{x_0}(\lambda) = \arg[\sigma(x_0, 1) - \lambda] - \arg[\sigma(x_0, -1) - \lambda]$.

Using the results of ⁽¹⁾, we obtain the following proposition:

Theorem. *The essential spectrum of the operator A consists of the values of the function $\sigma(x, \xi)$ for $x \in \Omega$, $\xi \in \Sigma$, and of those points λ for which $|d_{x_0}(\lambda)| \geq \pi$ ($|d_{x_0}^\pm(\lambda)| \geq \pi$) for some x_0 on the boundary of Ω .*

Example 1. Consider the operator

$$T : L_2(0, 1) \ni u(x) \rightarrow m(x)u(x) + \int_0^1 \frac{K(x, t)}{x - t} u(t) dt,$$

where $m(x)$ and $K(x, t)$ have continuous first derivatives. This operator was studied in ⁽²⁾. Write T in the form

$$T = A + C,$$

where

$$A : L_2(0, 1) \ni u(x) \rightarrow m(x)u(x) + k(x) \int_0^1 \frac{u(t)}{x - t} dt, \quad k(x) = K(x, x);$$

$$C : L_2(0, 1) \ni u(x) \rightarrow \int_0^1 \frac{K(x, t) - K(x, x)}{x - t} u(t) dt.$$

It is easy to see that C is a completely continuous operator. Consequently, the operators T and A have the same essential spectrum. Let

$$\mathcal{A} : L_2(-\infty, \infty) \ni u(x) \rightarrow m(x)u(x) + k(x) \int_{-\infty}^{\infty} \frac{u(t)}{x - t} dt.$$

Then

$$A = R_{\Omega} \mathcal{A} P_{\Omega},$$

where $\Omega = (0, 1)$. For the symbol of the operator $\mathcal{A}$ we obtain

$$\sigma(x, \xi) = m(x) + i\pi k(x) \operatorname{sgn} \xi.$$

The set of values of $\sigma(x, \xi)$ for $x \in (0, 1)$ and $|\xi| = 1$ consists of two curves

$$A_1 B_1 : m(x) + i\pi k(x), \quad 0 \leq x \leq 1,$$

$$A_2 B_2 : m(x) - i\pi k(x), \quad 0 \leq x \leq 1.$$

It is easy to see that the points λ for which $|d_{x_0}(\lambda)| \geq \pi$ fill two rectilinear segments $A_1 A_2$ and $B_1 B_2$, where

$$A_1 = m(0) + i\pi k(0), \quad A_2 = m(0) - i\pi k(0),$$

$$B_1 = m(1) + i\pi k(1), \quad B_2 = m(1) - i\pi k(1).$$

Thus, the essential spectrum of the operator T consists of the points of the curvilinear quadrilateral $A_1 B_1 A_2 B_2$. This agrees with the result ⁽²⁾, obtained with the aid of the theory of normed rings.

Example 2. Let $m \geq 2$, and let Ω be a bounded domain in m -dimensional Euclidean space with sufficiently smooth boundary $\partial\Omega$. Denote by $G(x; y)$, $x = (x_1, \dots, x_m)$, $y = (y_1, \dots, y_m)$, the Green's function of the problem

$$\Delta u = f, \quad u|_{\partial\Omega} = 0, \quad \Delta = \partial^2/\partial x_1^2 + \dots + \partial^2/\partial x_m^2.$$

Consider the operator

$$T : L_2(\Omega) \ni u(x) \rightarrow \int_{\Omega} \frac{\partial^2 G(x; y)}{\partial y_1^2} u(y) dy.$$

Differentiation under the integral is performed in the space of generalized functions. This operator was studied in ⁽³⁻⁵⁾.

It is not hard to see that T can be represented in the form

$$T = R_{\Omega} \mathcal{A} P_{\Omega} + C,$$

where C is a completely continuous operator, and

$$\mathcal{A} : L_2(E_m) \ni u(x) \rightarrow \begin{cases} \int_{E_m} \frac{\partial^2}{\partial y_1^2} (1 - r^{m-2}) u(y) dy, & \text{if } m > 2, \\ \int_{E_m} \frac{\partial^2}{\partial y_1^2} (\ln r) u(y) dy, & \text{if } m = 2. \end{cases}$$

The operator $\mathcal{A}$ has the symbol

$$\sigma(x, \xi) = \xi_1^2 |\xi|^{-2}.$$

Applying the theorem, we obtain that the essential spectrum of the operator T fills the interval $[0, 1]$.

Joint Institute for Nuclear Research

Received

2 IX 1969

REFERENCES

1. I. B. Simonenko, *Izv. AN SSSR, Ser. matem.*, **29**, Nos. 3 and 4 (1965).
2. J. Schwartz, *Comm. Pure and Appl. Math.*, **15**, 75 (1962).
3. S. L. Sobolev, *Izv. AN SSSR, Ser. matem.*, **18**, No. 1 (1954).
4. R. A. Aleksandryan, *Tr. Moskovsk. matem. obshch.*, **9**, 455 (1960).
5. R. D. Dentsev, *DAN*, **126**, No. 2 (1959).
6. R. Dentsev, *Izv. na Matem. inst. na Bolgarsk. Akad. nauk*, 1968.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.