

ON THE PURITY OF THE SET OF POINTS OF NON-SMOOTHNESS OF A MORPHISM OF SCHEMES

I. V. Dolgachev

1969

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196901.24462>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 513.013.7

MATHEMATICS

I. V. DOLGACHEV

ON THE PURITY OF THE SET OF POINTS OF NON-SMOOTHNESS OF A MORPHISM OF SCHEMES

(Presented by Academician L. S. Pontryagin on 24 II 1969)

1. Let $f : X \rightarrow Y$ be a locally finitely presented morphism of schemes. Denote by $\text{Sing}^X(f)$ the set of points $x \in X$ at which f is not smooth (recall that one of the equivalent definitions of smoothness of f at a point x consists in saying that f is flat at x and the fiber $f^{-1}(f(x))$ is geometrically regular at x). By virtue of ⁽³⁾, the set $\text{Sing}^X(f)$ is closed in X . Put also $\text{Sing}^Y(f) = f(\text{Sing}^X(f))$. If the morphism f is proper, then $\text{Sing}^Y(f)$ is closed in Y .

We say that f satisfies the purity theorem from above if, for every maximal point x_0 of the set $\text{Sing}^X(f)$ (i.e. the generic point of an irreducible component of $\text{Sing}^X(f)$), we have

$$\text{codim}(x_0, X) = \dim O_{X, x_0} \leq \dim_{x_0} f + 1.$$

Analogously, if f is proper, we say that f satisfies the purity theorem from below if, for every maximal point y_0 of the set $\text{Sing}^Y(f)$, we have

$$\text{codim}(y_0, Y) \leq 1.$$

Below we give several classes of morphisms of schemes satisfying the purity theorem from above (respectively, from below).

2. In what follows we shall always assume that all schemes under consideration are locally Noetherian, and that morphisms of schemes are locally of finite type.

If a morphism of schemes $f : X \rightarrow Y$ is smooth at a point $x \in X$, then the sheaf of relative differentials $\Omega_{X/Y}^1$ is free at the point x , and its rank at x is equal to the relative dimension $\dim_x f$ of the morphism f at the point x ⁽³⁾. Let us also note that f , moreover, is equidimensional at the point x , and consequently, if X and Y are integral, $\dim_x f$ coincides with the dimension of the generic fiber of f .

For the proof of the following criterion of smoothness, see ⁽³⁾.

Suppose that the sheaf $\Omega_{X/Y}^1$ is free at the point x of rank $\dim_x f$. Then f is smooth at x in any of the following cases:

A. f is flat at the point x .

B. f is a morphism of S -schemes, Y is smooth over S at the point $f(x)$, and the canonical homomorphism

$$f^*\Omega_{Y/S}^1 \rightarrow \Omega_{X/S}^1$$

is injective at the point x .

3. Suppose now that the schemes X and Y are integral. Assume that the morphism $f : X \rightarrow Y$ does not satisfy the purity theorem from above. Let x_0 be a maximal point of $\text{Sing}^X(f)$ for which

$$\text{codim}(x_0, X) > \dim_x f + 1.$$

Since x_0 is maximal on the set $\text{Sing}^X(f)$, f is smooth at every point $x \neq x_0$ such that $x_0 \in \overline{\{x\}}$. Considering the canonical embedding $j : \text{Spec } O_{X, x_0} \rightarrow X$, we obtain that the restriction of f to $\text{Spec } O_{X, x_0}$ is smooth at every point of the punctured local scheme $\overline{X}_{x_0} = \text{Spec } O_{X, x_0} - \{x_0\}$. In particular, the sheaf

$$j^*\Omega_{X/Y}^1 = \Omega_{\overline{X}/Y, x_0}^1$$

is locally free on the scheme $\overline{X}_{x_0}$. Moreover, if n denotes the dimension of the generic fiber of f , then the rank of the sheaf $j^*\Omega_{X/Y}^1$ at every point $x \in \overline{X}_{x_0}$ is equal to $\dim_x f = n$. Since the function $x \mapsto \dim_x f$ is upper semicontinuous ⁽³⁾, we obtain, moreover, that

$$n \leq \dim_x f < \text{codim}(x_0, X) - 1 = \dim O_{X, x_0} - 1.$$

Proposition. *Let A be an integral local ring, $\mathfrak{m}$ the maximal ideal of A , $X = \text{Spec } A$, $X' = X - \{\mathfrak{m}\}$. Let M be a finite A -module*

of finite projective dimension type ≤ 1 . Suppose that the restriction of the sheaf $L = M^\sim$ to X' is a locally free sheaf of rank $n < \dim A - 1$. Then M is a free A -module of rank n .

Proof. Suppose that M is not free and let

$$0 \rightarrow A^l \rightarrow A^m \rightarrow M \rightarrow 0$$

be its projective resolution (free, since A is local). Applying the functor $\text{Hom}(_, A)$, we obtain the exact sequence

$$A^m \rightarrow A^l \rightarrow \text{Ext}_A^1(M, A) \rightarrow 0.$$

Since $M^\sim$ is locally free on X , we have

$$\text{Supp}(\text{Ext}_A^1(M, A)) = \{\mathfrak{m}\}.$$

By ⁽¹⁾, Theorem 4, it follows that

$$\dim A \leq l - m + 1.$$

On the other hand, obviously $l - m = n$. Thus we obtain $n \geq \dim A - 1$, which contradicts the hypothesis of the lemma.

Hence, obviously, it follows

Theorem 1. Let $f : X \rightarrow Y$ be a morphism of locally finite type of integral locally Noetherian preschemes. Suppose that for every point $x \in X$ we have

$$\dim \operatorname{proj} \Omega_{X/Y, x}^1 \leq 1$$

and that one of the conditions A or B formulated above is satisfied. Then f satisfies the purity theorem above.

4. **Example 1.** Suppose that X and Y are smooth S -preschemes, and f is a morphism of S -preschemes smooth at the generic point of X (the latter condition means, of course, that f is dominant and that the corresponding extension $R(X)/R(Y)$ of fields of rational functions is separable). In this case we have an exact sequence of sheaves of relative differentials

$$0 \rightarrow f^* \Omega_{Y/S}^1 \rightarrow \Omega_{X/S}^1 \rightarrow \Omega_{X/Y}^1 \rightarrow 0,$$

see ⁽²⁾, Lemma 1. This shows that for every point $x \in X$,

$$\dim \operatorname{proj} \Omega_{X/Y, x}^1 \leq 1,$$

and, moreover, condition B is satisfied. Thus, by Theorem 1, the morphism f satisfies the purity theorem above.

The following example was pointed out to me by V. A. Danilov.

Example 2. The hypotheses of Theorem 1 are satisfied if $f : X \rightarrow Y$ is a flat morphism of integral schemes which is locally a complete intersection (for example, a quasiprojective flat morphism of regular schemes).

The following result is well known and is a special case of the Zariski-Nagata purity theorem ^(4,5).

Corollary. Suppose that f is a quasi-finite morphism of regular schemes. Then $\operatorname{Sing}^X(f)$ coincides with the set of points of X at which f is ramified, and defines a divisor on X .

5. Let now $f : X \rightarrow Y$ be a proper morphism. Whereas the question of purity above is local in X , the question of purity below for the morphism f is local in Y , which, obviously, requires consideration of the global properties of the morphism f .

In the case where f is a finite morphism of regular preschemes, the assertion that the purity theorem below holds for f is nothing other than the Zariski-Nagata purity theorem. The following result concerns the case of a morphism of relative dimension 1 (i.e. f is equidimensional with one-dimensional fibers).

Theorem 2. Let $f : X \rightarrow Y$ be a flat proper morphism of relative dimension 1 of integral regular schemes, whose generic fiber is a smooth geometrically connected curve of genus g . Suppose, in addition, that in the case $g > 1$, f is cohomologically flat in dimension 0 (i.e. $f_*(O_X) = O_Y$ universally), and in the case $g \geq 1$, for all $y \in Y$, $\text{char}(k(y)) = 0$. Then f satisfies the purity theorem below.

We note that in the case when f is separable (i.e., the fibers of f are geometrically reduced), the assertion of the theorem, without any additional conditions on f , follows trivially from Theorem 1 (Example 1). In the general case, however, the proof essentially uses the condition that the fibers of f are curves, and requires an interesting study of the properties of singular curves varying in a family.

Corollary. *Under the hypotheses of Theorem 2, if f is smooth over an open set $V \subset Y$ containing the points of codimension ≤ 1 , then f is smooth on all of X .*

6. The restrictions on the cohomological flatness of the morphism f in the case $g > 1$, and on the characteristic in the case $g \geq 1$, arise in the course of the proof of Theorem 2 and, as it seems to us, are not essential. More generally, it seems plausible that any flat proper morphism of relative dimension 1 of regular schemes satisfies the purity theorem below. As follows immediately from Theorem 1 (Example 2), this is true, in any case, if f is, in addition, separable and is a locally flat intersection.

In conclusion, I express my gratitude to V. A. Danilov for his constant interest in the work and to M. Raynaud, who pointed out errors in the original proof of Theorem 2.

Moscow State University
named after M. V. Lomonosov

Received
14 II 1969

REFERENCES

1. D. Buchsbaum, D. S. Rim, Bull. Am. Math. Soc., **69**, 382 (1963).
2. I. Dolgachev, A. Parshin, Mat. zametki, **4**, No. 5 (1968).
3. A. Grothendieck, Éléments de Géométrie Algébrique, Chap. IV, Publ. Math., I. H. E. S., Paris, Nos. 28, 32, 1967.

4. M. Nagata, Ill. J. Math., **3**, 319 (1959).

5. O. Zariski, Proc. Nat. Acad. Sci. U. S. A., **44**, 791 (1958).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.