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Abstract

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MATHEMATICS

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ON THE BIRATIONAL EQUIVALENCE OF LINEAR ALGEBRAIC GROUPS

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1. Let k be a field of characteristic zero, $\bar{k}$ its algebraic closure, and $\mathcal{G}$ the Galois group of the extension $\bar{k}/k$. Let X be an algebraic variety over the field k , i.e. a geometrically irreducible reduced scheme of finite type over the field k ; $\bar{X} = X \otimes_k \bar{k}$ is the scheme obtained from X by extension of the ground field. The group $\mathcal{G}$ acts naturally on $\bar{X}$ and on the objects determined by the scheme $\bar{X}$; in particular, the group $\text{Pic } \bar{X} = H^1(\bar{X}, \mathcal{O}_{\bar{X}}^*)$ is a $\mathcal{G}$ -module.

Every $\mathcal{G}$ -module isomorphic to a finite direct sum of modules of the form $\mathbf{Z}[\mathcal{G}] \otimes_{\mathbf{Z}[U]} \mathbf{Z}$, where U runs through some set of subgroups of finite index in the group $\mathcal{G}$, will be called, following Yu. I. Manin, trivial. If $\bar{X}$ is a nonsingular variety, then every submodule of the divisor group generated by an invariant finite set of prime divisors is trivial.

Two $\mathcal{G}$ -modules A and B will be called similar if there exist trivial $\mathcal{G}$ -modules M_1 and M_2 such that there is an isomorphism of direct sums $A + M_1 \simeq B + M_2$. The following assertion is apparently known; for surfaces it was proved in ⁽¹⁾, and over an arbitrary perfect field.

Theorem 1. *Let X and Y be nonsingular projective varieties over the field k , birationally equivalent over this field. Then the $\mathcal{G}$ -modules $\text{Pic } \bar{X}$ and $\text{Pic } \bar{Y}$ are similar.*

The proof is based on the existence of a nonsingular model Z and birational morphisms

$$X \xleftarrow{\varphi} Z \xrightarrow{\psi} Y$$

with subsequent use of Leray's spectral sequence for the maps φ and ψ .

Corollary 1. *If the variety X is rational, i.e. $\bar{X}$ is birational to $\bar{\mathbf{P}}^n$, then $\text{Pic } \bar{X}$ is a $\mathcal{G}$ -module without torsion of finite $\mathbf{Z}$ -rank.*

Corollary 2. *If the variety X is rational over the field k , i.e. X is birational to $\mathbf{P}^n$, then the $\mathcal{G}$ -module $\text{Pic } \bar{X}$ is similar to a trivial one; in particular, $H^1(k, \text{Pic } \bar{X}) = 0$.*

2. Let G be a connected linear algebraic group defined over a field k of characteristic zero. Consider some nonsingular projective variety V , defined over the field k , such that the variety $\text{Spec } k[G]$ is mapped biregularly onto an open subvariety of the variety V . We shall call the variety $V = V(G)$ a projective model of the group G over k .

It is easy to see that such models always exist (the field k has characteristic zero). Let $\bar{G} = \text{Spec } \bar{k}[G]$, and let $F = \bar{V} \setminus \bar{G}$ be a closed subset invariant with respect to the group $\mathcal{G}$. Considering the exact sequence of local cohomology of the sheaf $\mathcal{O}_{\bar{V}}^*$, defined by the embedding $\bar{G} \subset \bar{V}$, we obtain the exact sequence of $\mathcal{G}$ -modules

$$0 \rightarrow \hat{G} \rightarrow M \rightarrow \text{Pic } \bar{V} \rightarrow \text{Pic } \bar{G} \rightarrow 0, \quad (1)$$

where $\hat{G}$ is the $\mathcal{G}$ -module of rational characters of the group G , and M is a trivial $\mathcal{G}$ -module generated by divisors whose supports lie in F .

3. Let us compute $\text{Pic } \bar{G}$, or, more generally, $\text{Pic } G = \text{Pic}(\text{Spec } k[G])$.

Theorem 2. Let G be a connected solvable algebraic group over k . Then

$$\text{Pic } G = H^1(k, \hat{G}).$$

Theorem 3. Let G be a connected semisimple algebraic group defined over a field k ; let π be the fundamental group on which the Galois group $\mathfrak{G}$ acts naturally. Then

$$\text{Pic } G = \text{Hom}(\pi, \bar{k}^*)^{\mathfrak{G}}.$$

In particular, if the group G is simply connected, then

$$\text{Pic } G = 0.$$

It follows from these theorems that, for any algebraic linear group G , the group $\text{Pic } G$ is finite. The proof of Theorem 2 is based on the fact that $\text{Pic } \bar{G} = 0$ and the group $\text{Pic } G$ is computed as the kernel of the map $\text{Pic } G \rightarrow \text{Pic } \bar{G}$. Similarly, in Theorem 3 it is first proved that $\text{Pic } \bar{G} = 0$ for simply connected groups G , and then the kernel of the map $\text{Pic } G \rightarrow \text{Pic } \bar{G}$ is computed.

4. Let us return to the sequence (1). Let T be a torus defined over the field k . Then $\text{Pic } \bar{T} = 0$, and from (1) we obtain the exact sequence

$$0 \rightarrow \hat{T} \rightarrow \hat{M} \rightarrow \text{Pic } \bar{V} \rightarrow 0. \quad (1)$$

All $\mathfrak{G}$ -modules in the sequence (2) are, up to torsion, of finite rank, and the module M is trivial. If the torus T is rational over the field k , then the module $\text{Pic } \bar{V}$ is similar to a trivial one. Adding in this case a trivial module to the modules M and $\text{Pic } \bar{V}$, one can arrange that in the sequence (2) the modules M and $\text{Pic } \bar{V}$ will be trivial. Hence

Theorem 4. In order that the torus T be rational over the field of definition k , it is necessary that its character module $\hat{T}$ can be included in an exact sequence

$$0 \rightarrow \hat{T} \rightarrow \hat{M} \rightarrow \hat{N} \rightarrow 0,$$

where $\hat{M}$ and $\hat{N}$ are trivial $\mathfrak{G}$ -modules.

It would be interesting to determine whether this necessary condition for rationality is also sufficient.

Let now k be a number field, k_p its p -adic completion (p a finite or infinite place of the field k). Consider two sets of arithmetic origin associated with the group G over k . The first is $\mathfrak{W}(G)$, i.e. the kernel of the map

$$H^1(k, G(\bar{k})) \rightarrow \prod_p H^1(k_p, G(\bar{k}_p)).$$

The second is constructed as follows. Take the direct product $\prod_p G(k_p)$ with the direct-product topology. The group $G(k)$ is embedded in $\prod_p G(k_p)$ by means of the diagonal map. Let $\overline{G(k)}$ be the closure of $G(k)$ in $\prod_p G(k_p)$. Denote by $A(G)$ the quotient set

$$\prod_p G(k_p) / \overline{G(k)}.$$

The set $A(G)$ measures the failure of weak approximation in the group G .

Theorem 5. Let T be a torus defined over a number field k . Then there exists an exact sequence of groups

$$0 \rightarrow A(T) \rightarrow H^1(k, \text{Pic } \bar{V}) \rightarrow \mathfrak{W}(T) \rightarrow 0.$$

Corollary. If $H^1(k, \text{Pic } \bar{V}) = 0$, then

$$A(T) = \mathfrak{W}(T) = 0,$$

and conversely. In particular, if the torus T is rational over the field k , then $\mathfrak{W}(T) = 0$ and the Tamagawa number $\tau(T)$ is an integer.

The validity of the theorem on weak approximation in this case had been well known earlier.

Theorem 6. Let k be an arbitrary field of characteristic zero; let T be a torus over k that splits over a cyclic extension of the field k ; and let V be a projective model of the torus T . Then

$$H^1(k, \text{Pic } \bar{V}) = 0.$$

Corollary 1. Let T be a torus defined over a number field with cyclic splitting field. Then

$$A(T) = \mathfrak{W}(T) = 0.$$

These facts are known: the equality $\mathfrak{W}(T) = 0$ was proved by T. Ono [2], and the equality $A(T) = 0$ was proved by J.-P. Serre (unpublished).

Corollary 2. *Let T be a torus defined over the field of algebraic numbers and split over a field L whose decomposition groups are cyclic for all primes. Then $A(T) = 0$. (J.-P. Serre, unpublished).*

5. We shall show that there exist tori with cyclic splitting field which are not rational over the field of definition. Let L be a cyclic extension of a field k of degree 23; let H be the Galois group of L/k . It is known that there exists a projective H -module $\hat{T}$, not free, such that $\hat{T} \otimes Q \simeq Q[H]$. Suppose that the torus T is rational over the field k . Then, by Theorem 4, there exists an exact sequence of $\mathcal{G}$ -modules $0 \rightarrow \hat{T} \rightarrow \hat{M} \rightarrow \hat{N} \rightarrow 0$, where $\hat{M}$ and $\hat{N}$ are trivial $\mathcal{L}$ -modules. Simple computations show that then $\text{Ext}_{\mathcal{G}}(\hat{N}, \hat{T}) = 0$, i.e. $\hat{M} \simeq \hat{T} + \hat{N}$. It is proved that $\hat{M}$ and $\hat{N}$ may be regarded as H -modules. But trivial H -modules are Z and $Z[H]$, and Reiner's theorem 3 shows that from the equality $\hat{T} + \hat{N} = \hat{M}$ it follows that $\hat{T} \simeq Z[H]$. This contradiction shows that the torus T is not rational over k . It is known that one can choose projective nonfree H -modules $\hat{T}_1$ and $\hat{T}_2$ so that $\hat{T}_1 + \hat{T}_2 \simeq Z[H] + Z[H]$. Hence

$$T_1 \times T_2 \simeq R_{L/k}(G_m) \times R_{L/k}(G_m).$$

On the right is a direct product of tori rational over k . This gives an interesting example of the rationality of the direct product of two varieties which are not rational over k .

6. Let Q be the field of rational numbers, $K = Q(\sqrt{a}, \sqrt{b})$ an extension of degree 4. Consider the tori

$$T(a, b) = R_{K/Q}^{(1)}(G_m)$$

of dimension 3 for different fields K . Let V be a projective model of the torus T . Then

$$H^1(Q, \text{Pic } \bar{V}) = Z_2$$

for any field K .

Consider two cases:

- a) All decomposition groups of the field K are cyclic. Then

$$A(T) = 0, \quad (T) = Z_2.$$

- b) There exists a decomposition group of order 4. Take, for example, a prime number $p > 2$ such that the Legendre symbol

$$\left(\frac{-1}{p}\right) = -1.$$

There are infinitely many such primes p . Then

$$[Q_p(\sqrt{p}, \sqrt{-1}) : Q_p] = 4$$

and the tori $T(p, -1)$ and $T(p', -1)$, for different such p and p' , are birationally nonequivalent over the field Q , since the torus $T(p', -1)$ is rational over the field Q_p for all $p \neq p'$ and is not rational over the field $Q_{p'}$. This shows that in dimension 3 there is an infinite set of birationally nonequivalent tori over the field Q . In dimensions 1 and 2 all tori are rational over Q .

7. Let G be a connected semisimple algebraic group defined over a field k of characteristic zero. Since the character group $\hat{G}$ is trivial, the sequence (1) takes the form

$$0 \rightarrow M \rightarrow \text{Pic } \bar{V} \rightarrow \text{Pic } \bar{G} \rightarrow 0,$$

where M is a trivial $\mathcal{G}$ -module. A theorem, analogous in form to Theorem 5 for tori, is proved.

Theorem 7. *Let G be a connected semisimple algebraic group defined over a field of algebraic numbers k ; let π be the fundamental group of the group G ; let V be a projective model of the group G .*

Then there exists an exact sequence

$$0 \rightarrow A \rightarrow H^1(k, \text{Pic } \bar{V}) \rightarrow B \rightarrow 0,$$

where

$$A = \text{Coker} \left[H^1(k, \pi) \rightarrow \prod_p H^1(k_p, \pi) \right],$$

$$B = \text{Ker} \left[H^2(k, \pi) \rightarrow \prod_p H^2(k_p, \pi) \right] = \text{Ker} \left[H^1(k, \text{Pic } \bar{G}) \rightarrow \prod_p H^1(k_p, \text{Pic } \bar{G}) \right].$$

If for the simply connected covering $\tilde{G}$ of the group G the sets $A(\tilde{G})$ and $(\tilde{G})$ are equal to zero (there are conjectures that this is always so), then $A(G)$ —

an abelian group that is a subgroup of the group A . I have not been able to relate the group B to the set $\mathfrak{W}(G)$, as in the case of tori. However, the following assertion is true under the condition $A(\tilde{G}) = 0 = \mathfrak{W}(\tilde{G})$:

Corollary. If G is a semisimple group defined and rational over the field of algebraic numbers, then $\mathfrak{W}(G) = 0$.

Furthermore, the order of the group B enters the denominator in the formula obtained by T. Ono (4) for the ratio of the Tamagawa numbers of the groups G and $\tilde{G}$. In our notation it has the following form:

$$\tau(G)/\tau(\tilde{G}) = [\text{Pic } G]/[B].$$

Suppose that A. Weil's hypothesis holds, i.e. $\tau(\widetilde{G}) = 1$. Then

$$\tau(G) = [\text{Pic } G]/[B].$$

If the group G is rational over k , then $B = 0$ and

$$\tau(G) = [\text{Pic } G]$$

is an integer. It is curious that for tori T the Tamagawa number is expressed by a very similar formula

$$\tau(T) = [\text{Pic } T]/[\mathbb{W}(T)].$$

The corollaries of Theorems 5 and 7 make it possible to state the following conjecture (a necessary criterion for the rationality of algebraic groups).

Hypothesis. Let G be a connected algebraic linear group defined and rational over the field of algebraic numbers. Then

$$\mathbb{W}(G) = 0$$

and the Tamagawa number

$$\tau(G) = [\text{Pic } G].$$

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