

# ON A PERIODIC EQUATION OF SECOND ORDER

E. L. Tonkov

1969

SovietRxiv

---

View the original and related papers at <https://sovietrxiv.org/items/ru-196901.23851>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

**Abstract**

**Full Text**

UDC 517.917

*MATHEMATICS*

E. L. TONKOV

## ON A PERIODIC EQUATION OF SECOND ORDER

*(Presented by Academician I. M. Vinogradov on 15 V 1968)*

The present note continues the investigations published in <sup>(10, 16)</sup>.

Let  $u_0, u_1$  ( $u_0(0) = 1, \dot{u}_0(0) = 0, u_1(0) = 0, \dot{u}_1(0) = 1$ ) be a fundamental system of solutions of the equation

$$\mathcal{L}x \equiv \ddot{x} + q(t)\dot{x} + p(t)x = 0, \quad t \in (-\infty, \infty), \quad (1)$$

where  $q, p \in L[0, \omega]$  are  $\omega$ -periodic.

By the multipliers of equation (1) (see <sup>(18)</sup>, p. 98) we shall mean the roots of the characteristic equation

$$\lambda^2 - [u_0(\omega) + \dot{u}_1(\omega)]\lambda + \exp(-\omega I_q) = 0$$

$$\left( I_q = \frac{1}{\omega} \int_0^\omega q(t) dt \right).$$

It is easy to see that a real number  $\lambda$  is not a multiplier if and only if there exists a Green's function for the problem

$$\mathcal{L}x = 0, \quad x(\omega) = \lambda x(0), \quad \dot{x}(\omega) = \lambda \dot{x}(0), \quad (2)$$

i.e., if and only if (2) has the unique trivial solution  $x(t) \equiv 0$ .

Many questions in the theory of equation (1) reduce to the estimation of multipliers (see <sup>(14)</sup>). For example, this equation is asymptotically stable for  $t \in [0, \infty)$  if and only if  $I_q > 0$  and, for all  $\lambda, |\lambda| \geq 1$ , there exists a Green's function for problem (2).

As usual, we identify two functions that differ only on a set of measure zero, and, in order to shorten formulations, omit qualifications necessary when dealing with summable functions (the inequality holds almost everywhere, etc.).

1. As in <sup>(10, 16)</sup>, denote by  $\alpha = r_0(\alpha), r_1(\alpha), r_2(\alpha), \dots$  the successive zeros of a nontrivial solution of equation (1) having its first zero at  $t = \alpha$  ( $r_i < r_{i+1}$ ).

**Lemma 1.** *A Green's function of problem (2) exists for every  $\lambda < 0$  (there are no negative multipliers of equation (1)) if and only if, for all  $\alpha \in [0, \omega]$ , either  $\alpha + \omega < r_1(\alpha)$ , or, for some  $n = 0, 1, \dots$ ,*

$$r_{2n+1}(\alpha) < \alpha + \omega < r_{2n+3}(\alpha).$$

2. Consider problem (2) for  $\lambda > 0$ . We shall say that **condition (A)** is satisfied if problem (2) has no eigenfunction of constant sign, and **condition (B)** if (2) has no sign-changing eigenfunction. Thus, for the existence of a Green's function of problem (2), it is necessary and sufficient that conditions (A) and (B) be satisfied simultaneously.

Put

$$\rho = \frac{1}{\omega} \ln \lambda, \quad p^*(\rho, t) \equiv \rho^2 + \rho I_q + \frac{1}{4} I_q^2 + p(t) - \frac{1}{4} q^2(t) - \frac{1}{2} \dot{q}(t).$$

**Lemma 2.** *Let  $\lambda > 0$ . Condition (A) is satisfied if and only if, for some natural  $k$ , on the interval  $[0, k\omega]$  there exists a function*

*function  $v(t)$  with absolutely continuous first derivative such that  $v(k\omega) = v(0)$ ,  $\dot{v}(k\omega) \leq \dot{v}(0)$  ( $\geq 0$ ),  $\mathcal{L}_\rho v \equiv \mathcal{L}v + 2\rho\dot{v} + (\rho^2 + \rho q)v \geq 0$  ( $\mathcal{L}_\rho v \leq 0$ ), and either  $\mathcal{L}_\rho v \not\equiv 0$ , or  $\dot{v}(k\omega) \neq \dot{v}(0)$ .*

**Theorem 1.** Suppose that any one of the following conditions is satisfied:

- a)  $p \neq 0$  on  $[0, \omega]$ ,  $I_q \geq -\rho$ , and for all  $\alpha \in [0, \omega]$

$$I(\alpha) \equiv \int_\alpha^{\alpha+\omega} dt \int_\alpha^t \exp \left( - \int_s^t q(\tau) d\tau \right) ds \geq -\rho\omega;$$

- b)  $p^*(\rho, t) \in L[0, \omega]$ ,  $p^*(\rho, t) \neq 0$  on  $[0, \omega]$ ,

$$\int_0^\omega p^*(\rho, t) dt \geq 0.$$

Then for  $\lambda = \exp \rho\omega$  condition (A) is satisfied.

**Lemma 3.** In order that, for every  $\lambda > 0$ , condition (B) be satisfied, it is necessary and sufficient that, for all  $\alpha \in [0, \omega]$  and some  $n = 0, 1, \dots$ , the inequalities

$$r_{2n}(\alpha) < \alpha + \omega < r_{2n+2}(\alpha). \quad (3)$$

hold.

**Corollary 1.** Suppose that for all  $\alpha \in [0, \omega]$ ,  $\alpha + \omega < r_2^*(\alpha)$ . Suppose, moreover, that one of the following conditions is satisfied:

- a)  $p \neq 0$  on  $[0, \omega]$ ,  $I_q \geq 0$ , and for all  $\alpha \in [0, \omega]$ ,  $I(\alpha) \geq 0$ ;
- b)  $p^*(0, t) \in L[0, \omega]$ ,  $p^*(0, t) \neq 0$  on  $[0, \omega]$ ,  $I_q \geq 0$ ,

$$\int_0^\omega p^*(0, t) dt \geq 0.$$

Then the Green's function of problem (2) exists for all  $\lambda \geq 1$ .

**Remark.** If inequalities (3) are satisfied for  $n \geq 1$ , then for any  $\lambda > 0$  the Green's function of problem (2) exists.

- 3. To construct effective conditions guaranteeing the inequality  $\beta < r_1(\alpha)$ , it is convenient to use the well-known Wintner-Levinson nonoscillation criterion <sup>(21)</sup> (see also Corollary 2 of <sup>(1)</sup>), namely that  $\beta < r_1(\alpha)$  if and only if on  $[\alpha, \beta]$  there exists a function  $v(t) \geq 0$  such that  $\mathcal{L}v \leq 0$ , with  $\mathcal{L}v \not\equiv 0$  on  $[\alpha, \beta]$ . From Remark 2 to Theorem 1 of <sup>(3)</sup> we obtain the following oscillation criterion.

**Lemma 4.**  $r_1(\alpha) < \beta$  if and only if on  $[\alpha, \beta]$  there exists such a  $v(t)$ , possessing an absolutely continuous first derivative, that  $\dot{v} \geq 0$ ,  $v(\alpha) = v(\beta) = 0$ ,  $\mathcal{L}v \geq 0$ , with  $\mathcal{L}v \not\equiv 0$  on  $[\alpha, \beta]$ .

**Theorem 2.** Suppose that  $q(t) \geq -2m_1 \leq 0$  for  $t \in [\alpha, \alpha + T_1]$ ,  $q(t) \leq 2m_2 \geq 0$  for  $t \in [\alpha + T_1, \alpha + T_1 + T_2]$ ,  $p(t) \geq k > m_i^2$  ( $i = 1, 2$ ), where

$$T_i = \frac{1}{\sqrt{k - m_i^2}} \left( \pi - \operatorname{arccotg} \frac{m_i}{\sqrt{k - m_i^2}} \right), \quad i = 1, 2.$$

If  $\beta - \alpha > T_1 + T_2$ , then  $r_1(\alpha) < \beta$ .

For the proof of Theorem 2 it suffices to use Lemma 4, taking as the function  $v(t)$  the solution of the equation  $\ddot{x} + M(t)\dot{x} + kx = 0$ , where  $M(t) = -2m_1$  for  $t \in [\alpha, \alpha + T_1]$  and  $M(t) = 2m_2$  for  $t \in [\alpha + T_1, \alpha + T_1 + T_2]$ .

For  $q \equiv 0$ , Theorem 2 gives the well-known oscillation criterion of N. E. Zhukovsky <sup>(5)</sup> (see also <sup>(18)</sup>): if  $p(t) \geq \pi^2/(\beta - \alpha)^2$  ( $\neq$ ), then  $r_1(\alpha) < \beta$ .

Analogously, the following assertion is proved, close in content to Corollary 3 of <sup>(9)</sup> (see also <sup>(4,12,13,19)</sup>).

**Theorem 3.** Suppose that  $q(t) \leq 2m_1 \geq 0$  for  $t \in [\alpha, \alpha + T'_1]$ ,  $q(t) \geq -2m_2 \leq 0$  for  $t \in [\alpha + T'_1, \alpha + T'_1 + T'_2]$ ,  $p(t) \leq l < m_i^2$  ( $i = 1, 2$ ),

where

$$T'_i = \frac{1}{\sqrt{l - m_i^2}} \operatorname{arctg} \frac{m_i}{\sqrt{l - m_i^2}}, \quad i = 1, 2.$$

If  $\beta - \alpha < T'_1 + T'_2$ , then  $r_1(\alpha) > \beta$ .

From Theorems 2 and 3 we obtain

**Corollary 2.** Let  $|q(t)| \leq 2m$ ,  $m^2 < k \leq p(t) \leq l$ , and for some integers  $n_1, n_2$  ( $0 \leq n_1 < n_2$ )

$$\frac{2n_1}{\sqrt{k - m^2}} \left( \pi - \operatorname{arctg} \frac{m}{\sqrt{k - m^2}} \right) \leq \beta - \alpha \leq \frac{2n_2}{\sqrt{l - m^2}} \operatorname{arctg} \frac{m}{\sqrt{l - m^2}}.$$

Then

$$r_{n_1}(\alpha) < \beta < r_{n_2}(\alpha). \quad (4)$$

For other effective sufficient conditions guaranteeing inequalities (4), see <sup>(9,11,20)</sup>.

4. The linear assertions formulated above prove useful in the study of questions of existence, uniqueness, and stability of periodic solutions of the nonlinear equation

$$\mathcal{L}x \equiv \ddot{x} + q(t)\dot{x} + p(t)x = f(t, x, \dot{x}), \quad t \in (-\infty, \infty). \quad (5)$$

We shall assume that  $f(t, x, y)$  satisfies the Carathéodory smoothness conditions <sup>(6)</sup>, is  $\omega$ -periodic in  $t$ , and is nonnegative for  $x \geq 0$ ,  $y \in (-\infty, \infty)$ .

The following assertion supplements Theorem 1 of <sup>(7)</sup>.

**Theorem 4.** Let the Green's function of problem (2) for  $\lambda = 1$  exist and be positive. Suppose, further, that for  $t \in [0, \omega]$ ,  $x \geq 0$ ,  $y \in (-\infty, \infty)$ , the inequality

$$f(t, x, y) \leq a(t)x + b(t)|y|^\gamma + c, \quad 0 \leq \gamma < 1$$

is satisfied.

If the Green's function of the problem  $\mathcal{L}x = a(t)x$ ,  $x(\omega) = x(0)$ ,  $\dot{x}(\omega) = \dot{x}(0)$  exists and is positive, then equation (5) has at least one positive  $\omega$ -periodic solution.

**Remark.** Theorem 4 remains valid if  $f(t, x, y)$  is discontinuous in  $x, y$ , satisfies the conditions of <sup>(2)</sup>, and the solution is understood in the sense of <sup>(2)</sup>.

The following assertion supplements the results of <sup>(10)</sup> on conditions for positivity of the Green's function of problem (2) for  $\lambda = 1$ .

**Theorem 5.** Let the equation  $\mathcal{L}x = 0$  be nonoscillatory on  $[a, a + \omega]$ , i.e., for every  $a \in [0, \omega)$ ,  $a + \omega < r_1(a)$ . Suppose, further, that  $p \neq 0$  on  $[0, \omega]$  and at least one of the following conditions is fulfilled:

- a) for all  $a \in [0, \omega)$ ,  $I(a) \geq 0$ ;
- b) for all  $a \in [0, \omega)$ ,

$$\int_a^{a+\omega} p(t) \exp \left( \int_a^t q(s) ds \right) dt \geq 0;$$

- c)  $p^*(0, t) \in L[0, \omega]$ ,

$$\int_0^\omega p^*(0, t) dt \geq 0.$$

Then the Green's function  $G(t, s)$  of problem (2) for  $\lambda = 1$  exists and  $G(t, s) > 0$  in the square  $t, s \in [0, \omega]$ .

For other conditions guaranteeing positivity of the Green's function of problem (2) for  $\lambda = 1$ , see <sup>(8,15,17)</sup>.

For the case when the right-hand side of equation (5) does not depend on  $\dot{x}$ , i.e., for the equation

$$\mathcal{L}x \equiv \ddot{x} + q(t)\dot{x} + p(t)x = f(t, x), \quad t \in (-\infty, \infty), \quad (6)$$

the following assertion is valid.

**Theorem 6** (cf. (7)). Let the Green's function of problem (2) for  $\lambda = 1$  exist and be positive. Suppose, further, that for  $t \in [0, \omega]$  and  $x_1 \geq x_2 \geq 0$  the inequalities

$$b(t)(x_1 - x_2) \leq f(t, x_1) - f(t, x_2) \leq a(t)(x_1 - x_2)$$

hold. If the Green's function of the problem  $\mathcal{L}x = a(t)x$ ,  $x(\omega) = x(0)$ ,  $\dot{x}(\omega) = \dot{x}(0)$  exists and is positive, and for the equation  $\mathcal{L}x = b(t)x$  one has  $\alpha + \omega < r_2(\alpha)$  for all  $\alpha \in [0, \omega)$ , then equation (6) has a unique positive  $\omega$ -periodic solution.

I take this opportunity to express my deep gratitude to my supervisor, Prof. N. V. Azbelev, for his constant support and assistance in this work.

Tambov Institute  
of Chemical Engineering

Received  
14 V 1968

## REFERENCES

1. N. V. Azbelev, DAN, 102, No. 3 (1955).
2. N. V. Azbelev, Li Mun Sa, R. K. Ragimkhanov, DAN, 171, No. 2 (1966).
3. N. V. Azbelev, Tr. Tambovsk. inst. khim. mashinostr., vol. 2 (1968).
4. V. G. Boltyanskii, *Mathematical Methods of Optimal Control*, "Nauka," 1966.
5. N. E. Zhukovskii, Matem. sborn., 16, No. 3 (1892).
6. E. Kamke, *Lebesgue-Stieltjes Integral*, Moscow, 1959.
7. Yu. S. Kolesov, DAN, 172, No. 2 (1967).
8. Yu. S. Kolesov, A. Yu. Levin, *Problems of Mathematical Analysis of Complex Systems*, vol. 1, Voronezh Univ., 1967.
9. Yu. V. Komlenko, DAN, 174, No. 5 (1967).
10. Yu. V. Komlenko, E. L. Tonkov, DAN, 179, No. 1 (1968).
11. A. Yu. Levin, DAN, 145, No. 6 (1962).
12. G. N. Milshtein, Differents. uravn., 1, No. 12 (1965).
13. V. E. Mirakov, DAN, 170, No. 4 (1966).
14. V. M. Starzhinskii, PMM, 18, No. 4 (1954).
15. E. L. Tonkov, Tr. Tambovsk. inst. khim. mashinostr., vol. 1 (1967).
16. E. L. Tonkov, Tr. Tambovsk. inst. khim. mashinostr., vol. 2 (1968).
17. A. Ya. Khokhryakov, B. M. Arkhipov, Differents. uravn., 1, No. 3 (1965).
18. L. Cesari, *Asymptotic Behavior and Stability of Solutions of Ordinary Differential Equations*, IL, 1964.
19. H. Erheser, Math. Zs., 61, No. 4 (1955).

20. A. S. Galbraith, Proc. Am. Math. Soc., 17, No. 2 (1966).

21. Ch. J. de la Vallée Poussin, J. Math. pure et appl., 8 (9) (1929).

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*