

AN INVESTIGATION OF THE HERMITE-FEJÉR INTERPOLATION PROCESS

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Abstract

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MATHEMATICS

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AN INVESTIGATION OF THE HERMITE-FEJÉR INTERPOLATION PROCESS

(Presented by Academician S. N. Bernstein on November 6, 1968)

1°. Let a matrix of numbers be given

$$-1 \leq x_n^{(n)} < x_{n-1}^{(n)} < \dots < x_1^{(n)} \leq 1, \quad n = 1, 2, \dots \quad (m)$$

Denote by C the set of all functions $f(x)$ continuous on $[-1, 1]$. By $L_n(f, x)$ we shall denote the Lagrange interpolation polynomial of degree $(n-1)$, constructed for the function $f(x)$ and the n -th row of the matrix (m) .

According to the classical theorem of S. N. Bernstein–G. Fejér, there is no matrix of nodes (m) for which, for every $f \in C$, the relation $L_n(f, x) \rightarrow f(x)$, $n \rightarrow \infty$, holds uniformly on $[-1, 1]$.

In this connection the theorem of L. Fejér is of interest; it is formulated as follows. Let the matrix (m) be composed of the roots of the polynomials of P. L. Chebyshev $T_n(x) = \cos n \arccos x$, i.e.

$$x_k^{(n)} = \cos \frac{2k-1}{2n} \pi, \quad k = 1, 2, \dots, n; \quad n = 1, 2, \dots, \quad (1)$$

and let $H_n(f) \equiv H_n(f, x)$ be the polynomial of degree $(2n-1)$, constructed for $f \in C$ and the nodes (1), and uniquely determined by the conditions $H_n(f, x_k^{(n)}) = f(x_k^{(n)})$, $H_n'(f, x_k^{(n)}) = 0$, $k = 1, 2, \dots, n$. Then for every $f \in C$ the relation $H_n(f, x) \rightarrow f(x)$, $n \rightarrow \infty$, holds uniformly on $[-1, 1]$.

The interpolation process $\{H_n(f)\}_{n=1}^{\infty}$ is usually called the Hermite–Fejér interpolation process. It was the subject of important investigations by L. Fejér and his students.

2°. In ⁽¹⁾ it was found that if the matrix (m) is composed of the numbers

$$\begin{aligned} x_{n+1}^{(n+2)} &= -1; & x_k^{(n+2)} &= \cos \frac{2k-1}{2n} \pi, & k &= 1, 2, \dots, n; \\ x_0^{(n+2)} &= 1, & n &= 1, 2, \dots, \end{aligned} \quad (2)$$

then the interpolation process $\{H_n(f)\}_{n=1}^\infty$, constructed for $f(x) = |x|$, diverges at the point $x = 0$. Since the matrix (2) is obtained from the matrix of Chebyshev nodes by adding the points $x = \pm 1$ as nodes, in connection with Fejér's theorem this result seems unexpected. Quite naturally there arose the question whether there exists such a continuous function on $[-1, 1]$ for which the process $\{H_n(f)\}_{n=1}^\infty$, constructed at the nodes (2), diverges at all points of the interval $(-1, 1)$. The answer to this question is given by Theorem 1.

Theorem 1. *The interpolation process $\{H_n(f)\}_{n=1}^\infty$, constructed for $f(x) = 1 - x^2$ at the nodes (2), diverges at every point of the interval $(-1, 1)$.*

We outline the proof. For the nodes (2),

$$\begin{aligned} H_{n+2}(f, x) = & \frac{f(1)}{2} [1 - (2n^2 + 1)(x - 1)](x + 1)T'_n(x) \\ & - \frac{f(-1)}{2} [1 + (2n^2 + 1)(x + 1)](x - 1)T'_n(x) \\ & + \sum_{j=1}^n f(x_j) \left(1 + \frac{3x_j(x - x_j)}{1 - x_j^2} \right) \frac{4x_j(x^2 - 1)^2 T_n^2(x)}{(x^2 - x_j^2)(x_j^2 - 1)[T'_n(x_j)]^2}. \end{aligned}$$

Therefore, for $f(x) = 1 - x^2$, $n = 4p$,

$$H_{n+2}(f, x) = \frac{2(x^2 - 1)T_n^2(x)}{n^2} \sum_{j=1}^{2p} \frac{4x_j^4 - (2x^2 + 1)x_j^2 - x^2}{(1 - x_j^2)(x^2 - x_j^2)^2}.$$

Decomposing the fraction under the $\sum$ into partial fractions and taking into account the symmetry of the nodes, we obtain

$$H_{n+2}(f, x) = \frac{(1 - x^2)T_n^2(x)}{n^2} \left(\frac{3}{x^2 - 1} \sum_{j=1}^n \frac{1}{1 - x_j} + \sum_{j=1}^n \frac{1}{(x - x_j)^2} + \frac{3x}{1 - x^2} \sum_{j=1}^n \frac{1}{x - x_j} \right).$$

We now use the identities:

$$\sum_{k=1}^n \frac{1}{\sin^2 \theta_k/2} = 2n^2, \quad \theta_k = \frac{(2k - 1)\pi}{2n}, \quad (3)$$

$$\frac{T'_n(x)}{T_n(x)} = \sum_{j=1}^n \frac{1}{x - x_j}, \quad \frac{T_n^2(x)(1 - x^2)}{n^2} \sum_{j=1}^n \frac{1}{(x - x_j)^2} = 1 - \frac{\sin 2n\theta \cos \theta}{2n \sin \theta}, \quad x = \cos \theta.$$

Consequently,

$$H_{n+2}(f, x) = 1 - 3 \cos^2 n\theta + \frac{\sin 2n\theta \cos \theta}{n \sin \theta}, \quad x = \cos \theta. \quad (4)$$

Since the process is symmetric, we may assume that $0 \leq x < 1$. Suppose that at $\bar{x} \in [0, 1)$ the process (4) converges. Then

$$\lim_{n \rightarrow \infty} \sin^2 n\theta = 1 - \frac{\cos^2 \theta}{3}, \quad \cos \theta = \bar{x},$$

and this contradicts the following lemma.

Lemma. For any θ from $[0, \pi/2]$ one can find a sequence of natural numbers $n_1 < n_2 < \dots, n_k \rightarrow \infty, k \rightarrow \infty$, such that the equality

$$\lim_{k \rightarrow \infty} \sin^2 n_k \theta = 0$$

holds.

Theorem 2. The interpolation process $\{H_n(f)\}_{n=1}^\infty$, constructed for $f(x) = x^2$ at the nodes (2), diverges at every point of the interval $(-1, 1)$.

The proof follows from Theorem 1 and the equality

$$H_{n+2}(1 - z^2, x) = 1 - H_{n+2}(z^2, x).$$

In connection with Theorem 2 it is of interest to investigate the convergence of the process $\{H_n(f)\}_{n=1}^\infty$, constructed at the nodes (2), for the function $f(x) = x$.

Theorem 3. The process $\{H_n(f)\}_{n=1}^\infty$, constructed for $f(x) = x$ at the nodes (2), diverges at all points of $(-1, 1)$ except $x = 0$.

We outline the proof. For $f(x) = x$ the identity holds

$$R_n(x) \equiv x - H_{n+2}(f, x) = \sum_{k=0}^{n+1} (x - x_k) L_k^2(x),$$

where $\{L_k(x)\}$ are the fundamental Lagrange polynomials of the nodes (2). After elementary transformations we obtain that

$$R_n(x) = \frac{\sin 2n\theta \sin \theta}{2n} - \frac{3 \cos \theta \sin^2 \theta}{2} \cos^2 n\theta. \quad (5)$$

Hence, with the aid of the lemma formulated earlier, it follows that the process diverges at every point $x \neq 0$ of $(-1, 1)$. The convergence of the process for $x = 0$ is obvious, since for $x = 0$ the right-hand side of (5) is equal to zero.

3°. The results of item 2° show that extending the Chebyshev nodes by adding the points $x = \pm 1$ as nodes substantially worsens the behavior of the Hermite-Fejér interpolation process. The question arises: does this situation occur for every system of nodes? From the following theorems it is clear that this is not always so.

Theorem 4. *The interpolation process $\{H_n(f)\}_{n=1}^\infty$, constructed for $f(x) = x^2$ at the nodes*

$$x_k^{(n+2)} = \cos k\pi/(n+1), \quad k = 0, 1, \dots, (n+1); \quad n = 1, 2, \dots, \quad (6)$$

converges uniformly on $[-1, 1]$. Moreover,

$$|H_{n+2}(f, x) - x^2| \leq 4/(n+1). \quad (7)$$

We outline the proof. For the nodes (6),

$$\begin{aligned} H_{n+2}(f, x) = \frac{U_n^2(x)}{(n+1)^2} & \left\{ \frac{f(1)}{4} \left[1 - \left(1 + \frac{2}{3}n(n+2) \right) (x-1) \right] (x+1)^2 \right. \\ & + \frac{f(-1)}{4} \left[1 + \left(1 + \frac{2}{3}n(n+2) \right) (x+1) \right] (x-1)^2 \\ & \left. + \sum_{j=1}^n f(x_j) \left(1 - \frac{x_j}{x_j^2-1} (x-x_j) \right) \left(\frac{x^2-1}{x-x_j} \right)^2 \right\}, \end{aligned}$$

$$U_n(x) = \sin(n+1)\theta / \sin \theta, \quad x = \cos \theta.$$

It is obvious that it suffices to prove the theorem for $f(x) = 1 - x^2$. In this case

$$H_{n+2}(f, x) = \frac{(1-x^2)^2 U_n^2(x)}{(n+1)^2} \sum_{j=1}^n \frac{1 + xx_j - 2x_j^2}{(x-x_j)^2}. \quad (8)$$

Hence, after elementary transformations using the differential equation for the Chebyshev polynomials of the second kind, we obtain

$$\begin{aligned} H_{n+2}(f, x) &= \frac{1}{(n+1)^2} \left[\sin^2 n\theta - 2n \sin n\theta \cos(n+1)\theta \sin \theta + n^2 \sin^2 \theta \right], \\ x &= \cos \theta. \end{aligned} \quad (9)$$

Estimate (7) follows immediately from (9).

Remark. From (8) and (9) follows the identity

$$\begin{aligned} & \sin^2(n+1)\theta \sum_{j=1}^n \frac{1 + \cos \theta_j \cos \theta - 2 \cos^2 \theta_j}{(\cos \theta - \cos \theta_j)^2} = \\ & = n^2 - 2n \frac{\sin n\theta}{\sin \theta} \cos(n+1)\theta + \left(\frac{\sin n\theta}{\sin \theta} \right)^2, \quad \theta_j = \cos \frac{j\pi}{n+1}. \end{aligned}$$

Putting here $\theta = \pi/2$ and $n = 2m$, we obtain

$$\sum_{j=1}^{2m} \frac{1}{\cos^2 j\pi/(2m+1)} = 4m(m+1).$$

This identity is an analogue of the well-known identity (3) of M. Riesz ⁽²⁾.

Theorem 5. *The interpolation process $\{H_n(f)\}_{n=1}^\infty$, constructed for $f(x) \equiv x$ at the nodes (6), converges uniformly on $[-1, 1]$. Moreover,*

$$|H_{n+2}(f, x) - x| \leq 1/n.$$

The proof follows from the identity

$$x - H_{n+2}(f, x) = \frac{1}{2(n+1)^2} (\cos \theta \sin^2(n+1)\theta - (n+1) \sin 2(n+1)\theta \sin \theta).$$

This theorem should be compared with the result of G. Szegő ([4], p. 349), according to which the Hermite-Fejér process constructed for $f(x) \equiv x$ at the nodes

$$\left\{ \cos \frac{k\pi}{n+1} \right\}_{k=1}^n, \quad n = 1, 2, \dots,$$

diverges at the point $x = 1$.

Theorem 6. *The process $\{H_n(f)\}_{n=1}^\infty$, constructed for $f(x) = |x|$ at the nodes (6), converges at the point $x = 0$. Moreover,*

$$|H_{n+2}(f, 0)| \leq C \ln n/n.$$

This theorem should be compared with the result in ([1]), according to which the Hermite-Fejér process constructed for $f(x) = |x|$ at the nodes (2) diverges at the point $x = 0$.

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