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Abstract

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MATHEMATICS

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ON ESTIMATING THE COMPLEXITY OF NORMAL ALGORITHMS

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1. In considering questions connected with the construction of normal algorithms, it is often necessary to indicate a general method of construction, for every alphabet, of a normal algorithm associated with this alphabet by a certain relation. Such are, for example, the methods of constructing the reversing and doubling algorithms given in A. A. Markov's monograph (¹) (II. § 4.12, II. § 4.13). The reversing and doubling algorithms for words in an n -letter alphabet constructed in accordance with these instructions have complexity (length of representation) of order n^2 . The question arises: can every general method of constructing a normal algorithm over an n -letter alphabet be replaced by another method that gives the desired algorithm with complexity depending linearly on n ; in particular, can one indicate a method of construction, for an n -letter alphabet, of a universal algorithm of complexity of order n (the universal algorithm $\mathfrak{U}_n$, constructed by E. S. Orlovskii in (⁴), has complexity of order n^2)? Below this question is considered in a precise formulation.
2. Since we shall be speaking only about normal algorithms, we agree to regard the terms *algorithm* and *normal algorithm* as synonyms.

Following A. A. Markov (²), by the representation of an algorithm $\mathfrak{A}$ in an alphabet not containing the letters α, β, γ , we shall mean the word $\mathfrak{A}^i$ which is obtained if, in the scheme of the algorithm $\mathfrak{A}$, in all simple substitution formulas the arrow is replaced by the letter α , in all final substitution formulas the word $\rightarrow \cdot$ is replaced by the letter β , at the end of each formula the letter γ is appended, and the resulting words are written one after another in the order in which the corresponding formulas occur in the scheme $\mathfrak{A}$.

By the complexity of the algorithm $\mathfrak{A}$ we shall mean the length of the word $\mathfrak{A}^i$. We shall denote the complexity of the algorithm $\mathfrak{A}$ by the symbol $\mathfrak{A}\mathfrak{J}$.

We shall consider natural numbers as words in the one-letter alphabet $|$ ((¹), I. § 3.13).

By the symbol $\mathfrak{A}_\sigma^\chi$ we shall denote the algorithm whose scheme is obtained from

the closure scheme of the algorithm $\mathfrak{A}$, if every final substitution formula of the form $U \rightarrow \cdot V$ is replaced by the formula $U \rightarrow \chi V$ and then every substitution formula of the form $\rightarrow V$ is replaced by the formula $\varepsilon \rightarrow \varepsilon V$ (($\hat{1}$), III. § 2.2, III. § 3.2.1).

3. Let us fix a sequence $a_1, a_2, \dots, a_n, \dots$ of pairwise distinct letters, and let the letters $a, b, \alpha, \beta, \gamma, \delta$ be pairwise distinct and distinct from the letters of this sequence. We form a sequence of alphabets A_n , putting $A_0 \rightleftharpoons \Lambda$, $A_{n+1} \rightleftharpoons A_n a_{n+1}$.

For every word X formed from the letters $\alpha, \beta, \gamma, a_1, \dots, a_n, \dots$, its translation $[X^\tau]$ is defined in the following way: $[\alpha^\tau \rightleftharpoons aba]$, $[\beta^\tau \rightleftharpoons ab^2a]$, $[\gamma^\tau \rightleftharpoons ab^3a]$, $[a_i^\tau \rightleftharpoons ab^{3+i}a]$ ($i \geq 1$), $[X\xi^\tau \rightleftharpoons [X^\tau][\xi^\tau]$ (ξ is one of the letters $\alpha, \beta, \gamma, a_1, \dots, a_n, \dots$).

The notation of an algorithm $\mathfrak{A}$ in any one of the alphabets A_n can be defined as the translation of the word $\mathfrak{A}^i$. We shall denote the notation of the algorithm $\mathfrak{A}$ by $\{\mathfrak{A}\}$. Whatever the natural number n and the algorithm $\mathfrak{A}$ in the alphabet A_n may be, $\{\mathfrak{A}\}$ is a word in the alphabet ab .

A word Q in the alphabet ab will be called a word of type $\mathfrak{z}$ if there can be specified a natural number n and an algorithm $\mathfrak{A}$ in the alphabet A_n such that

$$Q \simeq \xi \mathfrak{A} \mathfrak{z}.$$

For every word Q of type $\mathfrak{z}$, define the number $N(Q)$ to be the greatest of the numbers j ($j \geq 0$) such that the word b^{3+j} occurs in Q . For every word Q of type $\mathfrak{z}$ and for every natural number $j \geq N(Q)$, by $\langle Q, j \rangle$ we denote the algorithm in the alphabet A_j whose notation is the word Q .

By a constructive sequence of normal algorithms (c.s.n.a.) we shall mean any normal algorithm $\mathfrak{A}$ that transforms any natural number into some word of type $\mathfrak{z}$.

4. **Theorem 1.** For every natural number n an algorithm $\mathfrak{U}_n$ over the alphabet $A_n ab \alpha \beta \gamma \delta$ can be constructed satisfying the following conditions:

- 1) For any word P in the alphabet A_n and for any algorithm $\mathfrak{A}$ in A_n ,

$$\mathfrak{U}_n(\mathfrak{A} \mathfrak{w} \delta P) \simeq \mathfrak{A}(P);$$

- 2) If P is a word in the alphabet A_n , $j \geq n$, and $\mathfrak{A}$ is an algorithm in the alphabet A_j that transforms every word in A_n to which it is applicable also into a word in the alphabet A_n , then

$$\mathfrak{U}_n(\xi \mathfrak{A} \mathfrak{z} \delta P) \simeq \mathfrak{A}(P).$$

- 3) If P is a word in the alphabet A_n , then

$$\mathfrak{U}_n : P \vdash [P^\tau \neg].$$

$$4) \mathfrak{U}_n \mathfrak{Z} = 10n + 515.$$

Returning to the question posed at the beginning of the note, we propose to regard Theorem 1 and the following Theorem 2 as the answer to it.

Theorem 2. For every c.s.n.a. $\mathfrak{R}$ there can be constructed a c.s.n.a. $\mathfrak{S}$ such that, for every natural number n :

1) the algorithms

$$\langle \mathfrak{R}(n), \max(N(\mathfrak{R}(n)), n) \rangle \quad \text{and} \quad \langle \mathfrak{S}(n), \max(N(\mathfrak{S}(n)), n) \rangle$$

are equivalent with respect to the alphabet A_n ;

2)

$$\langle \mathfrak{S}(n), \max(N(\mathfrak{S}(n)), n) \rangle \mathfrak{Z} = 11n + 432 + \mathfrak{R}_\xi^x \mathfrak{Z}.$$

The proof of Theorem 2 is based on the existence of a c.s.n.a. generating algorithms with the properties formulated in items 2), 3), 4) of Theorem 1.

Theorem 3. There can be constructed a c.s.n.a. $\mathfrak{R}$ satisfying the following condition: if n is a natural number and $\mathfrak{A}$ is an algorithm over the alphabet A_n equivalent to the algorithm

$$\langle \mathfrak{R}(n), \max(N(\mathfrak{R}(n)), n) \rangle$$

with respect to A_n , then

$$\mathfrak{A} \mathfrak{Z} \geq 5n - 5.$$

Theorem 4. Whatever the natural numbers n and m , it is not true that an algorithm $\mathfrak{A}$ in the alphabet A_n satisfying the following condition is impossible: there is no algorithm $\mathfrak{B}$ over the alphabet A_n , equivalent to the algorithm $\mathfrak{A}$ with respect to the alphabet A_n , and such that $\mathfrak{B} \mathfrak{Z} \leq m$.

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Note: Figure translations are in progress. See original paper for figures.

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