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**Abstract**

**Full Text**

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**MATHEMATICS**

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# THE NEMYTSKII OPERATOR IN SPACES GENERATED BY GENERALIZED FUNC- TIONS

*(Presented by Academician A. N. Tikhonov, 18 III 1969)*

**1°.** Let  $X$  be a space with a  $\sigma$ -finite complete nonatomic measure  $\mu$ , with  $0 \leq \mu X \leq \infty$ ; let  $S$  be the set of all measurable functions on  $X$  with values in  $\bar{R} = [-\infty, \infty]$ .

Fix in  $S$  two functions  $a$  and  $b$ , with  $a(x) < b(x)$  for every  $x$ . Let to each  $x$  there correspond an interval  $\Delta_x$  with endpoints  $a(x)$  and  $b(x)$ , which may or may not be contained in  $\Delta_x$ . We shall assume that the sets  $\{x : a(x) \in \Delta_x\}$  and  $\{x : b(x) \in \Delta_x\}$  are measurable. It follows from this that the set

$$X_u = \{x : u \in \Delta_x\}$$

is measurable for any  $u \in \bar{R}$ .

Let the function  $g(u, x)$ , with values in  $\bar{R}$ , be defined for  $x \in X$ ,  $u \in \Delta_x$ , be continuous in  $u$  for almost every  $x$ , and measurable in  $x$  on  $X_u$  for every  $u \in \bar{R}$ . These conditions (Carathéodory conditions) ensure the measurability of the superposition  $g(\varphi(\cdot), \cdot)$ , if  $\varphi \in S$  and  $\varphi(x) \in \Delta_x$  for almost every  $x$  <sup>(10)</sup>.

For any  $D \subset S$  put

$$D(\Delta) = \{\varphi \in D : \varphi(x) \in \Delta_x \text{ almost everywhere (a.e.)}\}.$$

It follows from the preceding that the Nemytskii operator  $h$ , generated by the function  $g$ ,  $h\varphi = g(\varphi(\cdot), \cdot)$ , acts from  $S(\Delta)$  into  $S$ .

The Nemytskii operator in various function spaces has been studied in many works (see, for example, <sup>(1-6)</sup>). In the present note it is studied in a broad class of spaces including the spaces  $\mathcal{L}^p$  with weight,  $0 < p \leq \infty$ , Orlicz spaces (including in the sense of Zăănen), Orlicz–Nakano spaces <sup>(11)</sup>, Musielak–Orlicz  $F$ -spaces <sup>(13,14)</sup>, and others. Conditions are studied for the action, boundedness, and continuity of the operator  $h$  in these spaces. It is interesting to note that certain properties of the spaces under consideration are obtained here as simple consequences of theorems on the Nemytskii operator. The spaces in question are

generated by the so-called pre-generalized functions and generalized functions. Their definitions follow below.

A function  $M(u, x)$ ,  $0 \leq u \leq \infty$ ,  $x \in X$ , with values in  $[0, \infty]$ , is called a pre-generalized function if for almost every  $x$  it is nondecreasing and left-continuous in  $u$  (in the topology of  $\bar{R}$ ), and for each  $u$  the function  $M(u, \cdot) \in S$ , with  $M(0, \cdot) \in \mathcal{L}(X)$ , i.e. summable on  $X$ . Put

$$I_M \varphi = \int_X M[|\varphi(x)|, x] d\mu, \quad P_M = \{\varphi \in S_f : I_M \varphi < \infty\},$$

$$P_M^\alpha = \{\varphi \in S_f : \alpha \varphi \in P_M\}, \quad L^M = \bigcup_{\alpha > 0} P_M^\alpha, \quad L_f^M = \bigcap_{\alpha > 0} P_M^\alpha,$$

where  $S_f$  is the set of a.e. finite functions from  $S$ , with functions coinciding a.e. being regarded as equal. It is not difficult to show that  $L^M$  is a vector space, and  $L_f^M$  is its subspace.

A pre-generalized function  $M$  is called a generalized function if  $M(0, x) = 0$  and  $M(\infty, x) > 0$  a.e. on  $X$ , while  $M(+0, x) = 0$  a.e. on  $\{x : d_M(x) > 0\}$ , where

$$d_M(x) = \sup\{u \in [0, \infty] : M(u, x) < \infty\}.$$

If  $M$  is a generalized function, then  $L^M$  may <sup>(9,13,14)</sup> be regarded as an  $F$ -space with  $F$ -norm  $\|\varphi\|_M =$

$$= \inf\{\varepsilon > 0 : I_M(\varepsilon^{-1} \varphi) \leq \varepsilon\}.$$

Moreover,  $\|\varphi_n\|_M \rightarrow 0$  if and only if  $I_M(\alpha \varphi_n) \rightarrow 0$  for every  $\alpha > 0$ .

2°. Let  $M$  and  $\Phi$  be pregenfunctions;  $A$  and  $B$  nonempty sets of positive numbers;

$$M_A = \bigcap_{\alpha \in A} P_M^\alpha,$$

$$\Phi^B = \bigcup_{\beta \in B} P_\Phi^\beta, \quad \delta_x = \Delta_x \cap \{u : |u| \sup A \leq d_M(x)\}.$$

We shall assume that  $M_A(\Delta)$  is nonempty. It is obvious that if  $\varphi \in M_A(\Delta)$ , then  $\varphi(x) \in \delta_x$  a.e. Consequently,  $\delta_x$  is nonempty for almost every  $x$ . In analogy with the set  $D(\Delta)$ , define  $D(\delta)$ . From the preceding it follows that  $M_A(\Delta) = M_A(\delta)$ .

**Theorem 1** (main). *The operator  $h$  acts from  $M_A(\Delta)$  into  $\Phi^B$ , i.e.  $h[M_A(\Delta)] \subset \Phi^B$ , if and only if there exist  $\alpha \in A$ ,  $\beta \in B$ ,  $\gamma \geq 0$ , and  $f \in \mathcal{L}(X)$  such that, for all  $x \in X$  and  $u \in \delta_x$ ,*

$$\Phi[\beta|g(u, x)|, x] \leq \gamma M[\alpha|u|, x] - f(x). \quad (1)$$

**Corollary 1.** *If  $h[M_A(\Delta)] \subset \Phi^B$ , then  $h[P_M^\alpha(\delta)] \subset P_\Phi^\beta$  for some  $\alpha \in A$ ,  $\beta \in B$ .*

This assertion can be strengthened. To this end put

$$W(\varphi_0, \alpha, \rho) = \{\varphi : I_M(\alpha\varphi) \leq I_M(\alpha\varphi_0) + \rho\}.$$

**Theorem 1'.** *If  $h[M_A(\Delta) \cap W(\varphi_0, \alpha_0, \rho_0)] \subset \Phi^B$ , where  $\varphi_0 \in M_A(\Delta)$ ,  $\alpha_0 \in A$ ,  $\rho_0 > 0$ , then  $h[P_M^\alpha(\delta)] \subset P_\Phi^\beta$  for some  $\alpha \in A$ ,  $\beta \in B$ .*

From Theorems 1 and 1', by varying the sets  $A$  and  $B$ , one can obtain various criteria for the action of the operator  $h$ . We list some of them.

- 1)  $h[P_M^\alpha(\Delta)] \subset L^\Phi$  if and only if, for some  $\beta > 0$ ,  $\gamma \geq 0$ , and  $f \in \mathcal{L}(X)$ , (1) holds for all  $x \in X$  and  $u \in \Delta_x$ . If

$$h[P_M^\alpha(\Delta) \cap W(\varphi_0, \alpha, \rho_0)] \subset L^\Phi,$$

where  $\varphi_0 \in P_M^\alpha(\Delta)$  and  $\rho_0 > 0$ , then

$$h[P_M^\alpha(\Delta)] \subset P_\Phi^\beta$$

for some  $\beta > 0$ .

Put

$$X^0 = \{x : 0 < d_M(x) < \infty\}.$$

- 2)  $h[L_M^f(\Delta)] \subset L^\Phi$  if and only if, for some  $\alpha > 0$ ,  $\beta > 0$ ,  $\gamma \geq 0$ , and  $f \in \mathcal{L}(X \setminus X^0)$ , (1) holds for all  $x \in X \setminus X^0$  and  $u \in \Delta_x$ , and

$$\Phi[\beta|g(0, \cdot)|, \cdot] \in \mathcal{L}(X^0).$$

If  $\mu X^0 = 0$  and

$$h[L_M^f(\Delta) \cap W(\varphi_0, \alpha_0, \rho_0)] \subset L^\Phi,$$

where  $\varphi_0 \in L_M^f(\Delta)$ ,  $\alpha_0 > 0$ , and  $\rho_0 > 0$ , then

$$h[P_M^\alpha(\Delta)] \subset L^\Phi$$

for some  $\alpha > 0$ .

If  $g(u, x) = u$  for  $x \in X$ ,  $u \in \Delta_x$ , then from Theorems 1 and 1' various embedding criteria follow.

- 3)  $P_M(\Delta) \subset P_\Phi$  if and only if there exist  $\gamma \geq 0$  and  $f \in \mathcal{L}(X)$  such that

$$\Phi(|u|, x) \leq \gamma M(|u|, x) + f(x)$$

for all  $x \in X$  and  $u \in \Delta_x$ .

- 4) If  $\mu X^0 = 0$  and

$$[L_M^f \cap \{\varphi : I_M(\alpha_0\varphi) \leq \rho_0\}] \subset L^\Phi,$$

where  $\alpha_0 > 0$  and  $\rho_0 > 0$ , then  $L^M \subset L^\Phi$ .

- 5)  $L^M \subset L^\Phi$  if and only if  $P_M \subset L^\Phi$ , i.e. if and only if, for some  $\beta > 0$ ,  $\gamma \geq 0$ , and  $f \in \mathcal{L}(X)$ ,

$$\Phi(\beta u, x) \leq \gamma M(u, x) + f(x)$$

for all  $u \geq 0$  and  $x \in X$ . This is a generalization of Theorem 1 from <sup>(11)</sup> (see also <sup>(12)</sup>, Lemma 3).

- 6)  $L^M = P_M$  if and only if  $P_M \subset P_M^2$ , i.e. if and only if there exist  $\gamma \geq 0$  and  $f \in \mathcal{L}(X)$  such that

$$M(2u, x) \leq \gamma M(u, x) + f(x)$$

for all  $u \geq 0$  and  $x \in X$  (cf. <sup>(4,12)</sup>).

3°. Let  $M$  be a genfunction, and let  $L^M$  be an  $F$ -space with  $F$ -norm  $\|\cdot\|_M$ . With the aid of Theorem 1 it is easy to determine the topological nature of the set  $P_M$  (cf. <sup>(7,8)</sup>).

Indeed, as is not difficult to show,  $\text{int } P_M = M^{(1,\infty)}$ . Consequently,  $P_M$  is open if and only if  $P_M \subset P_M^\alpha$  for some  $\alpha > 1$ , and this is equivalent to the coincidence of  $L^M$  with  $P_M$ .

Similarly,

$$\overline{P}_M = M^{(0,1)}.$$

Consequently,  $P_M$  is closed if and only if there exist  $\alpha \in (0, 1)$ ,  $\gamma \geq 0$ , and  $f \in \mathcal{L}(X)$  such that

$$M(u, x) \leq \gamma M(\alpha u, x) + f(x)$$

for all  $x \in X$  and  $u \in [0, d_M(x)]$ . Another criterion is also valid:  $P_M$  is closed if and only if  $P_M =$

$$= \{\varphi \in L^M : |\varphi(x)| \leq d_M(x) \text{ a.e.}\}.$$

Consequently, if  $\mu X^0 = 0$ , then  $P_M$  is closed if and only if  $L^M = P_M$ .

The theorems on boundedness of the Nemytskii operator proved in <sup>(6)</sup> for Orlicz spaces remain valid. Namely, let  $M$  and  $\Phi$  be generating functions. Then, if  $h(P_M) \subset L^\Phi$ , then

$$\sup\{\|h\varphi\|_\Phi : \|\varphi - \varphi_0\|_M \leq r\} < \infty,$$

where  $\varphi_0 \in L_M^f$  and  $0 < r < 1$  (if  $\varphi_0 = \theta$ , one may take  $r = 1$ ). Hence it follows that if  $h(L^M) \subset L^\Phi$ , then

$$\sup\{\|h\varphi\|_\Phi : \|\varphi\|_M \leq r\} < \infty$$

for every  $r > 0$ .

At the same time, if  $\mu X^0 = 0$  and  $h(L_M^f) \subset L^\Phi$ , but  $h$  does not act from  $L^M$  into  $L^\Phi$ , then there exists a set  $E \subset L_M^f$ , bounded in the  $F$ -norm  $\|\cdot\|_M$ , such that

$$\sup\{\|h\varphi\|_\Phi : \varphi \in E\} = \infty.$$

4°. Here we shall study conditions for continuity of the Nemytskii operator in spaces generated by generating functions. Depending on the choice of convergence in the spaces  $L^M$  and  $L^\Phi$ , different types of continuity are obtained. Below we introduce a definition of continuity which covers many special cases.

Let  $\varphi_0 \in L^M(\Delta)$ ,  $h\varphi_0 \in L^\Phi$ , where  $M$  and  $\Phi$  are generating functions;  $A$  and  $B$  are nonempty sets of positive numbers.

**Definition.** Suppose the following condition is satisfied: if

$$I_M[\alpha(\varphi_n - \varphi_0)] \rightarrow 0$$

for every  $\alpha \in A$ , where  $\varphi_n \in L^M(\Delta)$ ,  $n = 1, 2, \dots$ , then

$$I_\Phi[\beta(h\varphi_n - h\varphi_0)] \rightarrow 0$$

for some  $\beta \in B$ . Then we shall say that

$$h \in (A, B, \varphi_0).$$

Put

$$\begin{aligned} g_0(v, x) &= g(\varphi_0(x) + v, x) - g(\varphi_0(x), x), & h_0\psi &= g_0(\psi(\cdot), \cdot), \\ c_M(x) &= \max\{u : M(u, x) = 0\}, & p_\beta(v, x) &= \Phi[\beta|g_0(v, x)|, x]. \end{aligned}$$

Obviously, the function  $g_0$  is defined for  $x \in X$  and  $v \in \Delta_x^0 = \Delta_x - \varphi_0(x)$  (algebraic difference).

**Theorem 2.** If  $\sup A < \infty$  and  $\sup A \notin A$ , then  $h \in (A, B, \varphi_0)$  if and only if there exist  $\alpha \in A$ ,  $\beta \in B$ , such that

$$h_0[P_M^\alpha(\Delta^0)] \subset P_\Phi^\beta$$

and, a.e. on  $\{x : d_M(x) > 0\}$ , the condition is satisfied:

$$p_\beta(v, x) = 0 \quad \text{for } |v| \leq (\sup A)^{-1}c_M(x)$$

and

$$\lim p_\beta(v, x) = 0 \quad \text{as } |v| \rightarrow (\sup A)^{-1}c_M(x).$$

**Theorem 3.** If  $\sup A = \bar{\alpha} \in A$ , then  $h \in (A, B, \varphi_0)$  if and only if there exists  $\beta \in B$  such that

$$h_0[P_M^{\bar{\alpha}}(\Delta^0)] \subset P_\Phi^\beta$$

and, a.e.,

$$\lim p_\beta(v, x) = 0 \quad \text{as } M(\bar{\alpha}|v|, x) \rightarrow 0.$$

In <sup>(13,14)</sup> the concept of  $\rho$ -convergence in modular spaces was studied.

In the space  $L^M$ ,  $\rho$ -convergence of the sequence  $\varphi_n$  to  $\varphi_0$  ( $\varphi_n \xrightarrow{\rho} \varphi_0$ ) means that

$$I_M[\alpha(\varphi_n - \varphi_0)] \rightarrow 0$$

for some  $\alpha > 0$ . Consequently,  $\rho$ -continuity of the operator  $h$  at the point  $\varphi_0$  (i.e., if  $\varphi_n \xrightarrow{\rho} \varphi_0$ , then  $h\varphi_n \xrightarrow{\rho} h\varphi_0$ ) means that

$$h \in (\{\alpha\}, (0, \infty), \varphi_0)$$

for every  $\alpha > 0$ . Hence, from Theorem 3, it follows:

**Corollary 2.** The operator  $h$  is  $\rho$ -continuous at the point  $\varphi_0$  if and only if

$$h[L^M(\Delta)] \subset L^\Phi$$

and, for every  $\alpha > 0$ , there exists  $\beta > 0$  such that, a.e.,

$$\lim p_\beta(v, x) = 0 \quad \text{as} \quad M(\alpha|v|, x) \rightarrow 0.$$

In particular, if a.e.  $c_M(x) = 0$  and  $h[L^M(\Delta)] \subset L^\Phi$ , then the operator  $h$  is  $\rho$ -continuous on all of  $L^M(\Delta)$ .

With the aid of Theorem 3, the question of the relation between convergence in the  $F$ -norm ( $\|\varphi_n - \varphi_0\|_M \rightarrow 0$ ) and convergence in the mean ( $I_M(\varphi_n - \varphi_0) \rightarrow 0$ ) is also easily resolved.

**Corollary 3.** Convergence in the  $F$ -norm in the space  $L^M$  coincides with convergence in the mean if and only if  $L^M = P_M$  and, a.e.,

$$c_M(x) = 0$$

(cf. (4,8)).

**Theorem 4.** If  $\sup A = \infty$ , then  $h \in (A, B, \varphi_0)$  if and only if

$$h_0[P_M^\alpha(\Delta^0)] \subset P_\Phi^\beta$$

for some  $\alpha \in A$ ,  $\beta \in B$ .

**Corollary 4.** The operator  $h$  is continuous (in the sense of convergence in the  $F$ -norm) at the point  $\varphi_0$  if and only if, for every  $\beta > 0$ , there exists  $\alpha > 0$  such that

$$h_0[P_M^\alpha(\Delta^0)] \subset P_\Phi^\beta.$$

All results from (5) concerning continuity of the Nemytskii operator carry over to spaces generated by generating functions. We give two new propositions.

**Theorem 5.** If  $h[P_M(\Delta)] \subset L^\Phi$  and  $\sup A = \infty$ , then  $h \in (A, B, \varphi)$  for every  $\varphi \in P_M(\Delta)$  if and only if there exist  $\alpha \in A$ ,  $\beta \in B$ ,  $\gamma \geq 0$ , and  $f \in \mathcal{L}(X)$  such that, for all  $x \in X$ ,  $u', u'' \in \Delta_x$ ,

$$\Phi[\beta|g(u'', x) - g(u', x)|, x] \leq \gamma\{M(|u'|, x) + M(\alpha|u'' + u'|, x)\} + f(x).$$

**Theorem 6.** Let  $a \in L^M$  and  $b \in L^M$  (see 1<sup>0</sup>). If  $h[L^M(\Delta)] \subset L^\Phi$  and  $h$  is continuous on  $L^M(\Delta)$ , then  $h$  is uniformly continuous on  $L^M(\Delta)$ .

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