

AN EXTERNAL INVERSE PROBLEM FOR THE VOLUME POTENTIAL OF VARIABLE DENSITY FOR A BODY CLOSE TO A GIVEN ONE

A. I. Prilepko

1969

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196901.21163>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.944

MATHEMATICS

A. I. PRILEPKO

AN EXTERNAL INVERSE PROBLEM FOR THE VOLUME POTENTIAL OF VARIABLE DENSITY FOR A BODY CLOSE TO A GIVEN ONE

(Presented by Academician M. A. Lavrent'ev, 28 VI 1968)

1°. In this paper a solution is given of the inverse problem for the metaharmonic potential in the following formulation (^(1,7); see also ^(1,2,6-8)), where the question of uniqueness and stability of the external problem was investigated).

One seeks a body T_1 such that its external metaharmonic potential ($\chi \geq 0$) of a given variable density μ is equal outside the body T_1 to a given metaharmonic function H (H is a regular solution of the equation $\Delta H - \chi^2 H = 0$), H decreases at infinity as a metaharmonic potential and is close, in the sense of a certain functional metric, to the external metaharmonic potential V of the given body T of density μ . It is assumed that the boundary S of the body T belongs to the class $A^{(2,\lambda)}$.

In the present article, for the metaharmonic potential ($\chi \geq 0$), the existence and uniqueness of the solution of the indicated problem are proved. For the Newtonian potential ($\chi = 0$), in the case of constant density $\mu = 1$, a problem of this kind for stellar bodies was solved by V. K. Ivanov (⁽¹⁾), and under the assumption that the body T is a sphere, a similar problem was studied by L. N. Sretenskii (⁽⁸⁾).

We note that the main result obtained in this paper is new also for the Newtonian potential ($\chi = 0$): first, a general class of variable densities is considered; second, there are no restrictions of the "stellar" type on the body T . This remark also applies to the metaharmonic potential ($\chi \geq 0$), for which a similar problem was studied in the case of a constant density of a stellar body T in the author's work (⁽⁷⁾).

2°. By virtue of the smoothness conditions on the surface S , for sufficiently small v ($|v| \leq 3\varepsilon_0$, $\varepsilon_0 > 0$) every point $y = (y_1, y_2, y_3)$ of the three-dimensional space E^3 , by the formula

$$\mathbf{y} = \mathbf{x} + v\mathbf{n}_x \quad (1)$$

can be defined in a unique way in a neighborhood of the surface S with the aid of three curvilinear coordinates (ξ, η, v) , where $(\xi, \eta, 0)$ are the curvilinear coordinates of the point $\mathbf{x}$ of the surface S ; $\mathbf{n}_x$ is the unit vector of the exterior normal to the surface S at the point $\mathbf{x}$. Denote the metaharmonic potential ($\chi \geq 0$) of a set A , $A \subset E^3$, with density $\mu(y) \neq 0$ almost everywhere for $y \in A$, by

$$V(x; A, \mu) = \int_A \mu(y) \frac{e^{-\chi r_{xy}}}{r_{xy}} dy,$$

where $r_{xy} = |y - x|$ is the distance between the points y and x . Consider a bounded simply connected domain T , bounded by a surface S of class $A^{(2,\lambda)}$. Let the external metaharmonic potential $V(x; T, \mu)$ of the body T of density μ be known. In addition, suppose that outside a domain T_0 , lying inside T at a positive distance d from the boundary S , a metaharmonic function H is given which at infinity behaves like a metaharmonic potential.

In addition we shall assume that:

- 1) $\mu(y)$ is a given real analytic function in the domain D' ($D' \supset T + S$, the boundary of D' is at a positive distance, greater than $3\varepsilon_0$, from the surface S) and $\mu(y)$ is nowhere equal to zero on the surface S .
- 2) Each of the quantities

$$\left\| \frac{\partial H}{\partial \nu} - \frac{\partial V(T, \mu)}{\partial \nu} \right\|, \quad \left\| \frac{\partial^2 H}{\partial \nu^2} - \frac{\partial^2 V(T, \mu)}{\partial \nu^2} \right\|$$

does not exceed ωC , where $0 < \omega < d$, $C = C(T)$, $\omega = \omega(T, \mu, \varepsilon_0, d)$, and the norm $\|\cdot\|$ of the limiting, from outside T , values of

$$\frac{\partial}{\partial \nu}(H - V), \quad \frac{\partial^2}{\partial \nu^2}(H - V)$$

is equivalent to the norm in the space $C^{(1,\lambda)}(s)$.

Let $\{S_1\}$ be the class of surfaces whose equation in the curvilinear system of coordinates (1) has the form

$$\{v = \zeta(\xi, \eta)\}, \quad |v| \leq \varepsilon_0, \quad \zeta \in C^{(1,\lambda)}.$$

Under these conditions, for the body T , the surfaces S and S_1 , and the functions V and H , the following holds.

Theorem. *There exists, and moreover is unique, a surface S_1 bounding a body T_1 , satisfying the condition $\|\xi\| < d$, such that the external metaharmonic*

potential $V(x; T_1, \mu)$ of the body T_1 of the given density μ is equal to the given metaharmonic function H in the domain exterior to the surface S_1 , i.e.

$$H(x) = V(x; T_1, \mu) \quad \text{for } x \in E^3 \setminus \overline{T}_1.$$

3⁰. The derivation of the nonlinear integro-differential equation determining the boundary of the sought body T_1 is based on the ideas of V. K. Ivanov ⁽¹⁾, L. Lichtenstein ⁽³⁾, and A. M. Lyapunov ⁽⁴⁾. It should be noted that the arguments used in the theory of nonhomogeneous equilibrium figures ^(3,4), for our problem, lead to an integral equation of the first kind, which cannot be investigated even in the case $\mu = 1$, as is noted in ⁽¹⁾. Therefore, following the method of V. K. Ivanov ⁽¹⁾, we use the normal derivative of the potential, although for our case, owing to the variable density, the derivation of the principal nonlinear equation is carried out somewhat differently than in ⁽¹⁾.

Denote by D the domain bounded by the surface $S_{2\varepsilon_0}$, which in the curvilinear system of coordinates (1) is defined by the equation

$$v = 2\varepsilon_0.$$

The derivation of the equation and its investigation are based on the following

Lemma. *Let T be a simply connected domain bounded by a surface $S \in A^{(2,\lambda)}$; let $\hat{T}$ be a simply connected domain bounded by a surface $\hat{S}$, defined in the curvilinear system of coordinates (1) by the equation*

$$v = \zeta(\xi, \eta), \quad |v| \leq \varepsilon_0;$$

L be the domain with boundary $S_{2\varepsilon_0}$, D contain $T + S$ and $\hat{T} + \hat{S}$. There exist positive numbers Ω and K , depending only on the shape of T , such that for $\|\xi\| \leq \Omega$ one can construct a continuous one-to-one mapping of the form

$$\hat{y}_k = y_k + \alpha_k(y_1, y_2, y_3) \quad (k = 1, 2, 3)$$

of the domain D onto some domain $\hat{D}$, which maps $T + S$ onto $\hat{T} + \hat{S}$, and moreover the Jacobian $\hat{J}(y)$ of the mapping

$$\hat{J}(y) = \partial(\hat{y}_1, \hat{y}_2, \hat{y}_3) / \partial(y_1, y_2, y_3) \neq 0 \quad \text{for } y \in D;$$

the functions $\alpha_k(y) \in C^{(1,\lambda)}(D)$ ($k = 1, 2, 3$) and satisfy the condition

$$|\alpha_k|, |\partial\alpha_k/\partial y_i|, |\partial\alpha_k/\partial y_i|_\lambda \leq \Pi,$$

where Π is a positive number that may be taken equal to

$$K\|\xi\| = \Pi,$$

and, moreover, for the functions a_k the conditions are satisfied

$$a_k|_{\nu=0} = a_k(x) = \zeta(x)a_k(x), \quad x \in S; \quad \mathbf{n}_x = \{a_1, a_2, a_3\}; \quad \partial a_k / \partial \nu|_{\nu=0} = 0.$$

4°. Introduce the function

$$F_t^\varepsilon = \int_{T'_t} \mu(y_t) \frac{\partial}{\partial \nu} \left(\frac{e^{-\chi|y_t-z|}}{|y_t-z|} \right) dy_t \Big|_{\nu=t\zeta+\varepsilon},$$

where $0 < \varepsilon \leq \varepsilon_0$, the point z has curvilinear coordinates (ξ, η, ν) ($(\xi, \eta, 0)$ are the curvilinear coordinates of the point $x \in S$); T_t is the domain bounded by the surface S_t , depending on the parameter t , S_t being defined by the equation

$$\nu = t\zeta(\xi, \eta), \quad 0 \leq t \leq 1.$$

Expanding the function F_t^ε in a series in powers of t , putting $t = 1$, and letting ε tend to zero, we obtain for the function $\zeta(\xi, \eta)$, which determines the boundary of the sought body T_1 , the nonlinear integro-differential equation

$$A(\zeta\mu) = \frac{\partial}{\partial \nu} V(T, \mu) \Big|_{\nu=0} + \zeta \frac{\partial^2 V(T, \mu)}{\partial \nu^2} \Big|_{\nu=0} - \frac{\partial H}{\partial \nu} \Big|_{\nu=\zeta} + \Psi(\zeta), \quad (2)$$

where $\frac{\partial}{\partial \nu} V(T, \mu)$, $\frac{\partial^2 V(T, \mu)}{\partial \nu^2}$ are the exterior limiting values on the surface S of the metaharmonic potential $V(T, \mu) = V(x; T, \mu)$ of the body T with density μ ,

$$A(\zeta\mu) = 2\pi\zeta(x)\mu(x) - \int_S \frac{\partial}{\partial n_x} \frac{e^{-\chi r_{xy}}}{r_{xy}} \mu(y)\zeta(y) dS_y, \quad x \in S,$$

$$\Psi(\zeta) = \sum_{n=2}^{\infty} \left[\lim_{\varepsilon \rightarrow 0} \frac{1}{n!} \frac{d^n}{dt^n} F_t^\varepsilon \Big|_{t=0} \right].$$

We note that every sufficiently small solution of equation (2) is a solution of the problem.

For the operator $\Psi(\zeta)$ estimates hold that are similar to those proved in theorem 3 of the author's paper (7a) (see also (1, 7b)). In view of the indicated estimates, under the conditions of the main theorem the existence and uniqueness of a solution of equation (1) are proved, and this solution can be found by the method of successive approximations.

Remark. The posed problem has, moreover, a unique solution under the assumption that the body T consists of a finite number of simply connected finite

domains $T_1, \dots, T_m$ with boundaries $S_1, \dots, S_m$ of class $A^{(2,\lambda)}$ (with $T_k + S_k \neq T_j + S_j$, $j \neq k$); by S , the boundary of the body T , we denote the union of the boundaries $S_1, \dots, S_m$. Instead of the domain T_0 , one considers a body T^0 consisting of simply connected domains $T_1^0, \dots, T_m^0$, lying respectively inside $T_1, \dots, T_m$ at a positive distance d from $S_1, \dots, S_m$. The function H is a regular solution of the metaharmonic equation in the domain $E^3 \setminus \bar{T}$ with the corresponding decrease at infinity. It is assumed that $\mu(y)$ is the collection of given functions μ_k , analytic in the domains $D'_k = T_k + S_k$ ($k = 1, 2, \dots, m$), each function $\mu_k(y)$ being nowhere zero on the surface S_k .

Institute of Mathematics

Siberian Branch of the Academy of Sciences of the USSR

Received

31 V 1968

CITED LITERATURE

- ¹ V. K. Ivanov, *Izv. AN SSSR, ser. matem.*, **20**, 793 (1956).
- ² M. M. Lavrent'ev, *On Some Incorrect Problems of Mathematical Physics*, 1962.
- ³ L. Lichtenstein, *Figures of Equilibrium of a Rotating Fluid*, 1966.
- ⁴ A. M. Lyapunov, *Collected Works*, **5**, 1965.
- ⁵ I. P. Nedyalkov, *Izv. Geofiz. inst.*, **1**, 75 (1960); *DAN*, **144**, No. 4 (1962).
- ⁶ P. S. Novikov, *DAN*, **28**, No. 3, 465 (1938).
- ⁷ A. I. Prilepko, a) *DAN*, **154**, No. 3, 534 (1964); b) *Sibirsk. matem. zhurn.*, **6**, No. 6, 1332 (1965).
- ⁸ L. N. Sretenskii, *Izv. AN SSSR, ser. matem.*, **2**, 551 (1938).
- ⁹ A. N. Tikhonov, *DAN*, **39**, 195 (1943).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.