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Abstract

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MATHEMATICS

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POLYNOMIALS ORTHOGONAL OVER AREA AND BIEBERBACH POLYNOMIALS

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If in a simply connected domain G , lying in the finite part of the plane and bounded by a Jordan curve Γ , there is defined a nonnegative function $h(z)$, summable over the area of the domain G and differing from zero almost everywhere, then the system of polynomials $\{K_n(z)\}$, having positive leading coefficient and orthonormal over the area of the domain G with weight $h(z)$, is uniquely determined. The simplest properties of these polynomials in the case of an analytic contour and unit weight were first considered by T. Carleman ⁽¹⁾, while other cases and further properties were studied in the works ^(3,5,6,8). In the present note we set forth asymptotic and approximation properties of the polynomials $\{K_n(z)\}$ under various smoothness conditions on the contour and the weight, as well as estimates for the rate of convergence of Bieberbach polynomials to the mapping function in a closed domain, since Bieberbach polynomials are represented in terms of polynomials orthogonal over area.

Let the function $w = \Phi(z)$ map conformally and one-to-one the exterior D of the contour Γ onto the domain $|w| > 1$ under the conditions $\Phi(\infty) = \infty$ and $\Phi'(\infty) = \gamma > 0$. We shall say that a rectifiable Jordan curve Γ belongs to the class $C(p, \alpha)$, where p is a natural number and $0 < \alpha < 1$, if in the equation of the curve $z = \lambda(s)$, where s is arc length, the periodic function $\lambda(s)$ is continuously differentiable p times, and moreover $\lambda^{(p)}(s) \in \text{Lip } \alpha$.

§ 1. Let us first consider the case of unit weight.

Theorem 1. *If $\Gamma \in C(p+1, \alpha)$, where $p \geq 1$, then for the polynomials $\{K_n(z)\}$, orthonormal over the area of the domain G with weight $h(z) \equiv 1$, the formula holds*

$$K_n(z) = \sqrt{\frac{n+1}{\pi}} \Phi'(z) \Phi^n(z) \left[1 + O\left(\frac{\ln n}{n^{p+\alpha}}\right) \right], \quad z \in \overline{D}. \quad (1)$$

The estimate of the remainder term in formula (1) cannot be improved in order if one considers the set of curves of the class $C(p+1, \alpha)$ with a fixed constant in the Lipschitz condition for the last derivative. For curves of the class $C(1, \alpha)$, where $\alpha > 1/2$, the remainder term decreases at the rate $n^{1-2\alpha} \ln^2 n$. The

estimate of the remainder term is improved on a closed set F , located inside the domain D . Namely, if $\Gamma \in C(p+1, \alpha)$, where $p + \alpha > 1/2$, then for $z \in F \subset D$ the remainder term in formula (1) will decrease at the rate $n^{-p-\alpha-1}$. Under the same conditions $\Gamma \in C(p+1, \alpha)$, where $p + \alpha > 1/2$, for the leading coefficient λ_n of the orthonormal polynomial $K_n(z)$ the formula holds

$$\frac{\gamma^{2n+2}}{\lambda_n^2} = \frac{\pi}{n+1} \left[1 + O\left(\frac{1}{n^{2p+2\alpha}}\right) \right], \quad \gamma = \Phi'(\infty).$$

Let the function $f(z)$ be analytic in the domain G , continuous in the closed domain $\overline{G}$; let $\{a_n\}$ be its Fourier coefficients with respect to the system $\{K_n(z)\}$, and let $E_N(f, G)$ be the best uniform approximation of this function by polynomials of degree not exceeding N in the closed domain $\overline{G}$.

Theorem 2. If $\Gamma \in C(2, \alpha)$ and $h(z) \equiv 1$, then the inequality

$$\left| f(z) - \sum_{n=0}^N a_n K_n(z) \right| \leq c_1 E_N(f, \overline{G}) \ln N, \quad z \in \overline{G}, \quad (2)$$

holds, where the constant c_1 depends only on Γ .

If, however, $\Gamma \in C(1, \alpha)$, where $\alpha > 1/2$, then in the analogue of inequality (2), instead of $\ln N$ there will be the product $N^{1-\alpha} \ln N$.

§ 2. Let now the weight function have the form $h(z) = |\gamma(z)|^2$, where the function $\gamma(z)$ is analytic in the domain G , continuous and nonzero in the closed domain $\overline{G}$. Denote by $H_2(h, G)$ the set of functions analytic in the domain G whose modulus squared is integrable over the area of the domain G with weight $h(z)$, and let $E_N(f, H_2)$ be the best approximation of the function $f(z)$ by polynomials in the metric of the space $H_2(h, G)$. Further, we shall say that a function $f(z)$, analytic in the domain G , belongs to the class $W^{(p)}H^{(\alpha)}(\overline{G})$ if it is continuously differentiable p times in the closed domain $\overline{G}$, and $f^{(p)}(z) \in \text{Lip } \alpha$. For $p = 0$ the corresponding class will be denoted by $H_A^{(\alpha)}(\overline{G})$.

Theorem 3. If $\Gamma \in C(p+1, \alpha)$, where $p \geq 0$, and $\gamma(z) \in W^{(p)}H^{(\alpha)}(\overline{G})$, then for every function $f(z) \in H_2(h, G)$ the inequality

$$\left| f(z) - \sum_{n=0}^N a_n K_n(z) \right| \leq c_2(F) E_N(f, H_2) / N^{p+\alpha}, \quad z \in F \subset G,$$

holds uniformly inside the domain G , where the constant $c_2(F)$ contains in the denominator the distance from the closed set F to the contour Γ to the power $p+2$.

Theorem 4. If $\Gamma \in C(1, \alpha)$ and $\gamma(z) \in H_A^{(\alpha)}(\overline{G})$, then the estimate

$$|K_n(z)| \leq c_3 \sqrt{n+1}, \quad z \in \overline{G}.$$

holds.

Theorem 5. If $\Gamma \in C(1, \alpha)$ and $\gamma(z) \in H_A^{(\alpha)}(\overline{G})$, then for every function $f(z) \in W^{(p)}H^{(\alpha)}(\overline{G})$ the inequality

$$\left| f(z) - \sum_{n=0}^N a_n K_n(z) \right| \leq c_4 \ln N / N^{p+\beta-1/2}, \quad z \in \overline{G}.$$

holds.

Theorem 6. If $\Gamma \in C(p+1, \alpha)$, where $p + \alpha > 3/2$, and $\gamma(z) \in W^{(p)}H^{(\alpha)}(\overline{G})$, then every function $f(z)$ analytic in the domain G , whose primitive of order p is uniformly bounded in the domain G , is expandable in a series in the orthogonal polynomials $\{K_n(z)\}$, converging uniformly inside the domain G .

§ 3. In the more general case of a nonanalytic weight, the following results hold.

Theorem 7. If $\Gamma \in C(1, \alpha)$, and the weight function is merely bounded away from zero, i.e. satisfies the condition $h(z) \geq c_5 > 0$, then the estimates

$$|K_n(z)| \leq c_6(n+1), \quad z \in \overline{G},$$

$$\int_{\Gamma} |K_n(z)|^2 |dz| \leq c_7(n+1).$$

hold.

Theorem 8. If the positive function $h(z)$ satisfies in the closed domain $\overline{G}$ a Lipschitz condition of order $\alpha < 1$, and the contour Γ is a regular analytic curve, then the relations

$$|K_n(z)| \leq c_8 n \left(\frac{\ln n}{n} \right)^{\alpha/2}, \quad z \in \overline{G},$$

$$K_n(z) = \sqrt{\frac{n+1}{\pi}} g(z) \Phi^n(z) \left\{ 1 + O \left[\left(\frac{\ln n}{n} \right)^{\alpha/2} \right] \right\}, \quad z \in F \subset D,$$

hold.

where F is a closed subset of the domain D , and the function $g(z)$, analytic in the domain D , is determined by the boundary values of the function $h(z)$.

Theorem 9. If the weight function $h(z)$ is uniformly separated from zero, then the inequality

$$\left| f(z) - \sum_{n=0}^N a_n K_n(z) \right| \leq c_9 N E_N(f, \overline{G}), \quad z \in \overline{G}.$$

Theorem 10. If the function $f(z)$ is analytic in the domain G_R , bounded by the level line Γ_R , on which $|\Phi(z)| = R > 1$, and is continuous in the closed domain $\overline{G}_R$, then, under the conditions of Theorem 8, the inequality

$$\left| f(z) - \sum_{n=0}^N a_n K_n(z) \right| \leq c_{10}(R) \sqrt{N} E_N(f, \overline{G}_R), \quad z \in \overline{G}_R.$$

§ 4. Let us now consider Bieberbach polynomials. Let the function $w = \varphi(z)$ map the domain G conformally and one-to-one onto the disk $|w| < r_0$, under the conditions $\varphi(z_0) = 0$ and $\varphi'(z_0) = 1$, where z_0 is a fixed point of the domain G , and r_0 depends on z_0 . The Bieberbach polynomial $\pi_n(z)$, by definition, minimizes the integral

$$J(F_n) = \iint_G |F'_n(z)|^2 dx dy$$

on the set of polynomials $F_n(z)$ of degree n satisfying the same normalization conditions $F_n(z_0) = 0$ and $F'_n(z_0) = 1$.

The conditions and rate of convergence of Bieberbach polynomials in the closed domain $\overline{G}$ were first considered by M. V. Keldysh ⁽²⁾ and S. N. Mergelyan ⁽⁴⁾. In the present note we consider various cases of smoothness of the contour Γ .

Theorem 11. If $\Gamma \in C(1, \alpha)$, where $\alpha > 3/4$, then the inequality

$$|\varphi(z) - \pi_n(z)| \leq c_{11} E_n(\varphi, \overline{G}) \ln n, \quad z \in \overline{G}.$$

Theorem 12. If $\Gamma \in C(1, \alpha)$, where $\alpha > 1/2$, then the estimate

$$|\varphi(z) - \pi_n(z)| \leq c_{12} \ln n / n^{2\alpha}, \quad z \in \overline{G}.$$

Theorem 13. If $\Gamma \in C(1, \alpha)$, then the estimate

$$|\varphi(z) - \pi_n(z)| \leq c_{13} E_n(\varphi, \overline{G}) \sqrt{n} \ln n, \quad z \in \overline{G}.$$

Theorem 14. If the Jordan curve Γ has bounded curvature, then the inequality

$$|\varphi(z) - \pi_n(z)| \leq c_{14} \ln^2 n / n^2, \quad z \in \overline{G}.$$

Theorem 15. If $\Gamma \in C(p+1, \alpha)$, where $p \geq 1$, then for $0 \leq m \leq p$ the estimate

$$|\varphi^{(m)}(z) - \pi_n^{(m)}(z)| \leq c_{15} n^{1/2+m} / n^{p+\alpha}, \quad z \in \overline{G}.$$

The estimates of Theorems 11-14 strengthen the results of our note (7).

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