

ASYMPTOTIC BEHAVIOR OF THE TIME OF ATTAINMENT FOR SUMS OF RANDOM VARIABLES GOVERNED BY A REGULAR SEMI-MARKOV PROCESS

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Abstract

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MATHEMATICS

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ASYMPTOTIC BEHAVIOR OF THE TIME OF ATTAINMENT FOR SUMS OF RANDOM VARIABLES GOVERNED BY A REGULAR SEMI-MARKOV PROCESS

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Let $\xi_1(\alpha), \xi_2(\alpha)$ be, for each $\alpha \in [0, 1]$, a sequence of independent identically distributed random variables with $M\xi_1(\alpha) = a(\alpha)$ and $D\xi_1(\alpha) = b(\alpha)$. Introduce the random functional

$$\tau(\alpha, s) = \min \left(n : \sum_{k=1}^n \xi_k(\alpha) \geq s \right), \quad s > 0.$$

From the results given in ⁽¹⁾ it follows that in the case when:

- 1) $\lim_{\alpha \rightarrow 1} (\text{sl}) \xi_1(\alpha) = \xi_1(1)^*$;
- 2) $\lim_{\alpha \rightarrow 1} a(\alpha) = a(1) = 0$;
- 3) $\lim_{\alpha \rightarrow 1} b(\alpha) = b(1) \in (0, \infty)$.
- 4) there exists

$$\lim_{\alpha \rightarrow 1, t \rightarrow \infty} (1 + a(\alpha)\sqrt{t})^{-1} = q \in [0, 1],$$

$$\lim_{\alpha \rightarrow 1, t \rightarrow \infty} P \left\{ \frac{\tau(\alpha, w(\alpha, t))}{t} < z \right\} = \sigma(z) \sqrt{\frac{2}{\pi}} \int_{q/\sqrt{zb(1)}}^{\infty} \exp \left\{ \frac{1-q}{b(1)} - \frac{(1-q)^2}{2b^2(1)v^2} - \frac{v^2}{2} \right\} dv^{**},$$

where $w(\alpha, t) = \sqrt{t}/(1 - a(\alpha)\sqrt{t})$.

In the present paper this result is extended to a more general summation scheme, described below.

Let $T_1(\alpha) = \{\eta_\alpha(n), n = 0, 1, \dots\}$ be, for each $\alpha \in [0, 1]$, a homogeneous, ergodic Markov chain with a finite or countable set of states $H = \{1, 2, \dots, m\}$, $1 \leq m \leq \infty$, transition probability matrix $\|p_{ij}(\alpha)\|_{i,j=1}^m$ and stationary distribution $q_j(\alpha) > 0$, $j \in H$, and let $T_2(\alpha) = \{(\tau(\alpha, n, i), \gamma(\alpha, n, i)), n \geq 1, i \in H\}$ be, independent of $T_1(\alpha)$, a collection of random vectors independent in the aggregate such that: a) $\tau(\alpha, n, i) \in [0, \infty)$ with probability 1, $n \geq 1$, $i \in H$; b) the distributions $(\tau(\alpha, n, i), \gamma(\alpha, n, i))$, $i \in H$, do not depend on n .

Introduce the random functionals

$$\xi(\eta_0, \alpha, t) = \sum_{k=0}^{\nu(\alpha, t)} \gamma(\alpha, k),$$

* The notation $\lim_{\alpha \rightarrow \alpha'} (\text{sl}) \xi(\alpha) = \xi(\alpha')$ denotes weak convergence of distribution functions.

** $\sigma(z) = 0$ for $z < 0$; $\sigma(z) = 1$ for $z \geq 0$.

where

$$\nu(\alpha, t) = \max \left(n : \sum_{k=0}^n \tau(\alpha, k) \leq t \right), \quad \eta_0 = \eta_\alpha(0) = \text{const} \in H;$$

$$\tau(\alpha, n) = 0 \text{ for } n = 0; \quad \tau(\alpha, n) = \tau(\alpha, n, \eta_\alpha(n-1)) \text{ for } n \geq 1$$

$$\gamma(\alpha, n) = 0 \text{ for } n = 0; \quad \gamma(\alpha, n) = \gamma(\alpha, n, \eta_\alpha(n-1)) \text{ for } n \geq 1;$$

$$\tau(\eta_0, \alpha, s) = \inf(t : \xi(\eta_0, \alpha, t) \geq s), \quad s > 0.$$

Denote

$$\theta_0(\alpha) = \inf(t : \eta_\alpha(\nu(\alpha, t)) = i),$$

$$\theta_n(\alpha) = \inf(t : \nu(\alpha, t) > \nu(\alpha, \theta_{n-1}(\alpha)), \eta_\alpha(\nu(\alpha, t)) = i), \quad n \geq 1,$$

the times of successive entries of the chain $T_1(\alpha)$ into the state $i \in H$, and

$$(\tilde{\gamma}(\alpha, n, i), \tilde{\tau}(\alpha, n, i)) = \begin{cases} (0, 0), & \text{for } n = 0, \\ \left(\theta_n(\alpha) - \theta_{n-1}(\alpha), \sum_{k=\nu(\alpha, \theta_{n-1}(\alpha))+1}^{\nu(\alpha, \theta_n(\alpha))} \gamma(\alpha, k) \right), & \text{for } n \geq 1. \end{cases}$$

Obviously, the random vectors $(\tilde{\gamma}(\alpha, n, i), \tilde{\tau}(\alpha, n, i))$, $n \geq 1$, are independent and identically distributed, and

$$\begin{aligned} M \exp\{\sqrt{-1}(s\tilde{\gamma}(\alpha, 1, i) + t\tilde{\tau}(\alpha, 1, i))\} = \\ = g_\alpha(s, t, i) \left(\sum_{\substack{k \in H \\ k \neq i}} f_\alpha(s, t, k, i) p_{ik}(\alpha) + p_{ii}(\alpha) \right), \end{aligned}$$

where

$$g_\alpha(s, t, j) = M \exp\{\sqrt{-1}(s\gamma(\alpha, 1, j) + t\tau(\alpha, 1, j))\}, \quad j \in H;$$

$$f_\alpha(s, t, j, i) = M \exp \left\{ \sqrt{-1} \left(s \sum_{k=0}^{\Delta_i(\alpha)} \gamma(\alpha, k) + t \sum_{k=0}^{\Delta_i(\alpha)} \tau(\alpha, k) \right) \mid \eta_\alpha(0) = j \right\},$$

$$j \in H, \quad j \neq i,$$

$$\Delta_i(\alpha) = \min(n : \eta_\alpha(n) = i),$$

and the functions $f_\alpha(s, t, j, i)$, $j \in H$, $j \neq i$, satisfy the system of linear equations

$$f_\alpha(s, t, j, i) = g_\alpha(s, t, j) \left(\sum_{\substack{k \in H \\ k \neq i}} f_\alpha(s, t, k, i) p_{jk}(\alpha) + p_{ji}(\alpha) \right), \quad j \in H, \quad j \neq i,$$

and if the corresponding moments for the random variables $\tau(\alpha, 1, i)$, $\gamma(\alpha, 1, i)$, $i \in H$, exist, then it is not difficult to find the quantities

$$a_1(\alpha) = q_i(\alpha) M \tilde{\gamma}(\alpha, 1, i), \quad a_2(\alpha) = q_i(\alpha) D \tilde{\gamma}(\alpha, 1, i),$$

$$b(\alpha) = q_i(\alpha) M \tilde{\tau}(\alpha, 1, i).$$

Theorem. Suppose that for $T_j(\alpha)$, $j = 1, 2$, the following conditions are satisfied:

(A): 1. $\lim_{\alpha \rightarrow 1} p_{ij}(\alpha) = p_{ij}(1)$, $i, j \in H$.

$$2. \lim_{u \rightarrow \infty} \limsup_{\alpha \rightarrow 1} P\{\Delta_i(\alpha) > u \mid \eta_\alpha(0) = j\} = 0.$$

$$(B): 1. \lim_{\alpha \rightarrow 1} (d)(\tau(\alpha, 1, i), \gamma(\alpha, 1, i)) = (\tau(1, 1, i), \gamma(1, 1, i)), \quad i \in H.$$

$$2. \limsup_{\alpha \rightarrow 1} M\tau(\alpha, 1, j) < \infty.$$

$$3. \lim_{\alpha \rightarrow 1} M\tau(\alpha, 1, i) = M\tau(1, 1, i), \quad i \in H.$$

$$4. \limsup_{\alpha \rightarrow 1} M(\gamma, 1, j)^2 < \infty.$$

$$5. \lim_{\alpha \rightarrow 1} M(\gamma(\alpha, 1, i))^j = M(\gamma(1, 1, i))^j, \quad i \in H, \quad j = 1, 2.$$

$$(C): 1. \text{ For } \alpha > \alpha_0, \quad a_1(\alpha) \geq 0.$$

$$2. \quad a_1(1) = 0, \quad a_2(1), \quad b(1) \in (0, \infty).$$

It is not difficult to show that, under conditions (A) and (B),

$$\lim_{\alpha \rightarrow 1} a_j(\alpha) = a_j(1), \quad j = 1, 2, \quad \lim_{\alpha \rightarrow 1} b(\alpha) = b(1).$$

3. There exists

$$\lim_{\alpha \rightarrow 1, t \rightarrow \infty} (1 + a_1(\alpha)\sqrt{t})^{-1} = g \in [0, 1].$$

Then

$$\begin{aligned} & \lim_{\alpha \rightarrow 1, t \rightarrow \infty} P\left\{\frac{\tau(\eta_0, \alpha, w(\alpha, t))}{b(1)t} < z\right\} = \\ & = \sigma(z) \sqrt{\frac{2}{\pi}} \int_{q/\sqrt{za_2(1)}}^{\infty} \exp\left\{\frac{1-q}{a_2(1)} - \frac{(1-q)^2}{2a_2^2(1)v^2} - \frac{v^2}{2}\right\} dv, \end{aligned}$$

where

$$w(\alpha, t) = \sqrt{t}/(1 + a_1(\alpha)\sqrt{t}).$$

Remark 1. Let $\gamma(\alpha, n, i) = \xi(\alpha, n, i, \tau(\alpha, n, i))$, $n \geq 1$, $i \in \bar{H}$, where $\{\{\xi(\alpha, n, i, t), t \geq 0\}, n \geq 1, i \in \bar{H}\}$, independently of $T_1(\alpha)$ and $\{\tau(\alpha, n, i), n \geq 1, i \in \bar{H}\}$, is a collection of mutually independent, stochastically continuous homogeneous processes with independent increments and finite-dimensional distributions not depending on n . Then condition (B) will take the form:

$$1. \lim_{\alpha \rightarrow 1} (d) \tau(\alpha, 1, i) = \tau(1, 1, i), \quad i \in \bar{H}.$$

2. $\lim_{\alpha \rightarrow 1} \sup_{j \in \bar{H}} M(\tau(\alpha, 1, j))^2 < \infty.$
3. $\lim_{\alpha \rightarrow 1} M(\tau(\alpha, 1, i))^j = M(\tau(1, 1, i))^j, \quad i \in \bar{H}, \quad j = 1, 2.$
4. $\lim_{\alpha \rightarrow 1} (d) \xi(\alpha, 1, i, 1) = \xi(1, 1, i, 1), \quad i \in \bar{H}.$
5. $\lim_{\alpha \rightarrow 1} \sup_{j \in \bar{H}} M(\xi(\alpha, 1, i, 1))^2 < \infty.$
6. $\lim_{\alpha \rightarrow 1} M(\xi(\alpha, 1, i, 1))^j = M(\xi(1, 1, i, 1))^j, \quad i \in \bar{H}, \quad j = 1, 2.$

Remark 2. If one introduces the functionals

$$\hat{\xi}(\eta_0, \alpha, t) = \sum_{k=0}^{\nu(\alpha, t)} \gamma(\alpha, k) + \xi(\alpha, \nu(\alpha, t) + 1, \eta_\alpha(\nu(\alpha, t)), t - \theta_{\nu(\alpha, t)}(\alpha)),$$

$$\hat{\tau}(\eta_0, \alpha, s) = \inf\{t : \hat{\xi}(\eta_0, \alpha, t) \geq s\}, \quad s > 0,$$

then for $\hat{\tau}(\eta_0, \alpha, s)$ the result of the theorem holds if, in addition to conditions (A), (B), (C), the condition

$$\lim_{u \rightarrow \infty} \lim_{\alpha \rightarrow 1} \sup_{x \in [0, \infty)} \sup_{j \in \bar{H}} P\{\tau(\alpha, 1, j) > u + x \mid \tau(\alpha, 1, j) > x\} = 0.$$

is fulfilled.

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References

1. A. V. Skorokhod, N. P. Slobodyanyuk, *Limit Theorems for Random Walks*, Kyiv, 1969.

Note: Figure translations are in progress. See original paper for figures.

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