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Abstract

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MATHEMATICS

M. Sh. BIRMAN

SOME APPLICATIONS OF THE LOCAL CRITERION FOR THE EXISTENCE OF WAVE OPERATORS

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In note ⁽¹⁾ the author proposed a fairly general "local" criterion for the existence of wave operators. Here we indicate some consequences of this criterion, giving convenient sufficient conditions for the existence of wave operators "as a whole," and also give applications to a number of scattering problems for differential operators.

1. Let H^0, H be self-adjoint operators in a Hilbert space $\mathfrak{H}$. Below, $\mathfrak{D}(H), E(\cdot), P$ denote respectively the domain, the spectral measure, and the projector onto the absolutely continuous subspace for H . The notations $\mathfrak{D}(H^0), E^0(\cdot), P^0$ have the analogous meaning for H^0 . If the wave operators

$$W_{\pm}(H, H^0) = \text{s-lim}_{t \rightarrow \pm\infty} \exp(itH) \exp(-itH^0) P^0, \quad (1)$$

exist, then the scattering operator $S = W_+^* W_-$ commutes with H^0 . Let $P^0 \mathfrak{H}$ be decomposed into a direct integral, where the operator $H^0 P^0$ becomes the operator of multiplication by the independent variable λ . In this representation the operator S corresponds to "multiplication" by the operator-function S_{λ} .

If the operators $W_{\pm}(H, H^0)$ exist and, for almost all λ , S_{λ} differs from the identity operator by a nuclear operator*, then the ordered pair H^0, H will be called **subnormal**. If, in addition, the operators $W_{\pm}(H^0, H)$ exist, then the pair H^0, H will be called **normal**. We note that for a normal pair the operators $H^0 P^0$ and HP are unitarily equivalent.

We shall agree to say that the operator H^0 is **subordinate** to the operator H if, for some continuous functions φ, φ^0 such that $|\varphi^0(\lambda)| \rightarrow \infty$ and $|\varphi(\lambda)| \rightarrow \infty$ as $|\lambda| \rightarrow \infty$, one has

$$\mathfrak{D}(\varphi(H)) \subseteq \mathfrak{D}(\varphi^0(H^0)).$$

Theorem 1. Let the orthoprojectors M_k^0 commute with H^0 and $\sup_k M_k^0 = I$. The pair H^0, H is subnormal if, for any $k = 1, 2, \dots$, the operators HM_k^0 are defined on $\mathfrak{D}(H^0)$ and

$$HM_k^0 - H^0 M_k^0 \in \mathfrak{S}_1.$$

Theorem 2. The pair H^0, H is normal if the operator H^0 is subordinate to H and, for any finite interval δ ,

$$HE^0(\delta) - H^0 E^0(\delta) \in \mathfrak{S}_1.$$

Theorem 3. The pair H^0, H is normal if the operators H^0 and H are subordinate to each other and, for any finite interval δ ,

$$HE(\delta)E^0(\delta) - E(\delta)H^0 E^0(\delta) \in \mathfrak{S}_1.$$

From a technical point of view, a convenient means of verifying the conditions of Theorems 1–3 is provided by the criteria for nuclearity of integral operators obtained in ^(2,3).

* The class of nuclear operators is denoted below by γ_1 , that of completely continuous operators by γ_∞ .

2. Below, $a(x)$ (with various indices) will always denote bounded functions of the point $x \in R^m$ satisfying, for some $\beta > m/4$, the condition $a(x)(1 + |x|^2)^\beta \in L_2(R^m)$. As usual, $D_k = i\partial/\partial x_k$, $D^\omega = D_1^{\omega_1} \dots D_m^{\omega_m}$.

Let $\mathfrak{H} = L_2(R^m)$ and $H^0 = (-\Delta)^l$. The formally self-adjoint expression

$$(-\Delta)^l + \sum_{|\omega| \leq s} a_\omega(x) D^\omega$$

naturally generates a symmetric operator H' on $\mathfrak{D}(H^{0r})$, if $2lr \geq s$. Suppose that H' has self-adjoint extensions, and let H be any one of them.

Theorem 4. *The pair H^0, H is subnormal. The pair H^0, H is normal if*

$$(H^0 u, u) \leq c[(H' u, u) + (u, u)] \quad (u \in \mathfrak{D}(H^{0r}))$$

and H is the Friedrichs extension of the operator H' .

Note that the pair H^0, H is certainly normal if $s < 2l$. This strengthens the results of I. V. Stankevich [4], who imposed considerably more stringent requirements on $a_\omega(x)$. It is of interest, however, that nonnegative perturbation terms of arbitrarily high order may be introduced into H^0 .

3. Perturbations of order not higher than $2l$ may be introduced into the operator $H^0 = (-\Delta)^l$ by means of bilinear forms, which requires no smoothness of the coefficients. Let

$$V[u, u] = \int \sum_{|\sigma| \leq l, |\rho| \leq l} a_{\sigma\rho}(x) (D^\sigma u \overline{D^\rho u} + D^\rho u \overline{D^\sigma u}) dx \quad (2)$$

and let the terms in (2) corresponding to $|\sigma| = |\rho| = l$ form a nonnegative form. The quadratic form

$$\|\sqrt{H^0}u\|^2 + V[u, u] \quad (u \in \mathfrak{D}(\sqrt{H^0}))$$

is semibounded and closed. It corresponds to a certain self-adjoint operator H .

Theorem 5. *The pair H^0, H is normal.*

We note that, under the assumptions of items 2 and 3, the requirement of boundedness of the coefficients could be weakened, and in some cases removed altogether. Furthermore, Theorems 4 and 5 remain valid if one sets $H^0 = P(D)$, where $P(\xi)$ is an arbitrary real elliptic polynomial.

4. The ellipticity requirement for the unperturbed operator is not necessary for the applicability of Theorems 1-3. Let $P(\xi)$ be a real polynomial. We shall call a polynomial $Q(\xi)$ *admissible* for $P(\xi)$ if, for every $\alpha > 0$,

$$\overline{Q(\xi)}Q(\xi) \leq \alpha P^2(\xi) + C(\alpha). \quad (3)$$

It is clear that in this case the degree of Q must be less than the degree of P .

Along with the operator $H^0 = P(D)$, consider the self-adjoint operator H generated by the formally self-adjoint expression

$$P(D) - \sum_{\tau} a_{\tau}(x) Q_{\tau}(D). \quad (4)$$

Theorem 6. *The pair H^0, H is subnormal if the polynomials $Q_{\tau}(\xi)$ are admissible for $P^r(\xi)$ for some $r \geq 1$. The pair H^0, H is normal if $r = 1$ and*

$$\lim_{|\xi| \rightarrow \infty} |P(\xi)| = \infty. \quad (5)$$

Condition (5) singles out a class of operators $P(D)$, called in ⁽⁵⁾ strongly Carleman (s. c.) operators. This class contains hypoelliptic operators. For $m > 1$, s. c. operators are semibounded. On this basis one could introduce, as in Theorem 4, perturbations of higher orders that give a normal pair. One can also indicate an analogue of Theorem 5.

Let us note that condition (5) also admits a weakening. Thus, for example, by Theorem 3, a normal pair in $L_2(R^2)$ is formed by the operators $P(D)$, $P(D) + a(x)$ for $P(\xi) = \xi_1^4 - \xi_2^4$ (cf. ⁽⁶⁾), where the scattering problem is considered by Friedrichs' method for $P(\xi) = \xi_1^2 - \xi_2^2$.

5. The results of Sec. 4 carry over to the case of matrix differential expressions. Let $P(\xi)$ be a polynomial Hermitian $(n \times n)$ -matrix. Definition (3) of admissibility of the matrix $Q(\xi)$ for $P(\xi)$ retains its meaning if $Q(\xi)$ is understood to be the Hermitian adjoint matrix. We shall call the matrix $P(\xi)$ strongly Carleman (s. c.) if its eigenvalue $p(\xi)$ least in absolute value satisfies condition (5). Of course, an elliptic matrix is always an s. c. matrix.

Let $H^0 = P(D)$, and let the self-adjoint operator H be generated by a formally self-adjoint expression of the form (4), where $a_\tau(x)$ and $Q_\tau(\xi)$ are certain $(n \times n)$ -matrices.

Theorem 7. *The pair H^0, H is subnormal if all $Q_\tau(\xi)$ in (3) are admissible for $P_\tau(\xi)$ for some $r \geq 1$. The pair H^0, H is normal if $r = 1$ and $P(\xi)$ is an s. c. matrix.*

Hence, in particular, it follows that under perturbation of a Hermitian matrix $P(D)$ by an operator $a(x)$, there always exist wave operators (1).

As examples of application of the results of Secs. 2-5 one may cite quantum scattering problems for a nonrelativistic particle in a potential field, in an electromagnetic field, with spin-orbit interaction taken into account, etc., as well as the problem of scattering by an external field of a relativistic electron (the Dirac system). In all these cases a normal pair arises. The indicated problems have been studied in a number of works by individual methods (see, for example, (7, 8)). However, here too the information about the spectral properties of the scattering matrix S_λ is new.

6. In a number of questions, scattering problems naturally arise for pairs of operators acting in different spaces. One can obtain an analogue of the local criterion (1) for the existence of wave operators in this case. Here we shall give only the simplest consequences of the corresponding local criterion.*

Let the self-adjoint operators H^0, H act in the spaces $\mathfrak{H}^0, \mathfrak{H}$, respectively, and let the bounded and boundedly invertible operator J map $\mathfrak{H}^0$ into $\mathfrak{H}$. Following T. Kato (9), define the wave operators

$$W_\pm(H, H^0; J) = s\text{-}\lim_{t \rightarrow \pm\infty} \exp(itH)J \exp(-itH^0)P^0.$$

We shall call the triple H^0, H, J **subnormal** if there exist operators (6) isometric on $P^0\mathfrak{H}^0$, and the scattering matrix S_λ differs from the identity operator by a nuclear one. We shall call the triple H^0, H, J **normal** if, in addition, there exist operators $W_\pm(H^0, H; J^*)$ isometric on $P\mathfrak{H}$. In the latter case the absolutely continuous parts of the operators H^0 and H are unitarily equivalent.

Theorem 8. *Let the orthoprojectors M_k^0 commute with H^0 and $\sup_k M_k^0 \neq I^0$. The triple H^0, H, J is subnormal if, for any $k = 1, 2, \dots$, the operators HJM_k^0*

are defined on $\mathfrak{D}(H^0)$,

$$HJM_k^0 - JH^0M_k^0 \in \mathfrak{S}_1, \quad (J^*J - I^0)M_k^0 \in \mathfrak{S}_\infty.$$

* The results of Sec. 6 were obtained jointly with A. L. Belopol'skii.

Theorem 9. The triple H^0, H, J is normal if $J\mathfrak{D}(H^0) = \mathfrak{D}(H)$ and, for every finite interval δ ,

$$HJE^0(\delta) - JH^0E^0(\delta) \in \mathfrak{S}_1, \quad (J^*J - I^0)E^0(\delta) \in \mathfrak{S}_\infty.$$

7. Let b_0 be a constant positive definite $(n \times n)$ -matrix; let $b(x)$ be another matrix, with $0 < c_1 \leq b(x) \leq c_2$, and let $a(x) = b(x) - b_0$ satisfy the condition introduced in Sec. 2. On the set of n -component vector functions $u(x)$, in addition to the usual scalar product in $L_2(R^m)$, introduce the scalar products $[u, v]_0 = (b_0 u, v)$, $[u, v] = (bu, v)$. These define two new Hilbert spaces $\mathfrak{H}^0, \mathfrak{H}$, which are metrically equivalent to $L_2(R^m)$. Let $P(\xi)$ be a polynomial Hermitian matrix, $H^0 = b_0^{-1}P(D)$, $H = b^{-1}P(D)$, and let J be the identity operator. The operators H^0 and H are self-adjoint in $\mathfrak{H}^0$ and $\mathfrak{H}$, respectively.

Theorem 10. The triple H^0, H, J is subnormal. If $b_0^{-1}P(\xi)$ is a s.c. matrix, then the triple H^0, H, J is normal.

Let $P(\xi)$ be a linear homogeneous function of ξ . For one special class of such matrices, Wilcox⁽¹⁰⁾ (see also Kato⁽⁹⁾) proved (under a somewhat weaker decrease of $a(x)$) the existence of the isometric operators (6). The more difficult question of the existence of the operators $W_\pm(H^0, H; J^*)$ remained open. The first assertion of Theorem 10 shows that the existence of the operators (6) is not, in general, connected with any special properties of the matrix $P(\xi)$. The situation is otherwise with the second assertion of Theorem 10. The requirement that $b_0^{-1}P(\xi)$ be strongly Carleman means Petrovskii hyperbolicity for the corresponding system with time $b_0 u_t = P(D)u$. This condition is not satisfied, for example, for Maxwell's system. The latter can, however, be considered separately, at least for the case of an isotropic medium.

Theorem 11. Let $u = \{u_1, u_2\}$, where u_1, u_2 are three-component functions in R^3 , $P(D)u = \{\text{rot } u_2, -\text{rot } u_1\}$. Let $b_0, b(x)$ be scalars and let $\text{grad } b(x)$ be locally bounded. Then the operators H^0, H, J defined above form a normal triple.

We note that Theorem 11 is not a consequence of Theorem 9, but is proved in a similar way on the basis of the general local criterion for the existence of wave operators.

Leningrad State University
named after A. A. Zhdanov

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