

CUBATURE FORMULAS WITH PRESCRIBED DERIVATIVES IN THE PERIODIC CASE

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Abstract

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MATHEMATICS

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CUBATURE FORMULAS WITH PRESCRIBED DERIVATIVES IN THE PERIODIC CASE

(Presented by Academician S. L. Sobolev, 19 V 1969)

In the present paper, the results of S. L. Sobolev on the integration of periodic functions from ⁽¹⁻⁶⁾ are generalized to the case of cubature formulas that are n -dimensional analogues of the Hermite quadrature formula.

Recall that a function $f(x)$ is called periodic with period matrix H , $|H| = 1$, if for any integer vector B and all x one has $f(x) = f(x + H\beta)$.

Identifying in Euclidean space E_n points that differ from one another by a period, we obtain the n -dimensional torus Ω_0 . This torus is called the fundamental domain. The condition of fundamentality can be written in the form

$$\sum_{\beta} \mathcal{E}_{\Omega_0}(x - H\beta) = 1.$$

Here $\mathcal{E}_{\Omega_0}(x)$ is the characteristic function of the domain Ω_0 .

To each function $\varphi(x) \in K$ from the basic space ^(1,8) one can associate the function $\hat{\varphi}(x) \in \hat{K}$, defined on the torus, by the formula

$$\hat{\varphi}(x) = \varphi(x) * \Phi^{(0)}(x) = \sum_{\beta} \varphi(x + \beta),$$

where

$$\Phi^{(0)}(x) = \sum_{\beta} \delta(x + \beta).$$

If $l(x)$ is a periodic functional in E_n , then one can associate with it a functional $\hat{l}(x)$, defined on the torus by the formula

$$\int_{E_n} l(x) \varphi(x) dx = \int_{\Omega_0} \hat{l}(x) \hat{\varphi}(x) dx.$$

These most important concepts are set forth fully, for example, in ⁽¹⁾.

In our case the error functional has the form

$$l^{(t)}(x) = 1 - \sum_{|\alpha| \leq 2t} (-1)^{|\alpha|} C^{(\alpha)} \Phi_H^{(\alpha)}(x);$$

$$(l^{(t)}, \varphi) = \int_{\Omega_0} \varphi(x) dx - \sum_{|\alpha| \leq 2t} C^\alpha D^\alpha \varphi(0); \quad (1)$$

$$2m > n; \quad 2t \leq m - [n/2] - 1; \quad (l^{(t)}, 1) = 0. \quad (2)$$

Here

$$\Phi_H^{(\alpha)}(x) = \sum_{\beta} D^\alpha \delta(x - H\beta); \quad \Phi_H^{(\dot{0})}(x) = \sum_{\beta} \delta(x - H\beta); \quad |\alpha| = \alpha_1 +$$

$$+ \alpha_2 + \dots + \alpha_n; \quad D^\alpha = \partial^{|\alpha|} / \partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_n^{\alpha_n}; \quad C^{(\alpha)} \text{ are the coefficients of the cu-}$$

curacy formula. From $(l^{(t)}(x), 1) = 0$ it follows that $C^{(0)} = 1$. The following norms in $\tilde{L}_2^m(H)$ are used:

$$\tilde{L}_2^m(H) = \left\{ f : f(x) = \sum_{\beta} f_{\beta} e^{2\pi i(\beta H^{-1}, x)}, \quad \|f\|_{\tilde{L}_2^m}^2 = (2\pi)^{2m} \sum_{\beta} |f_{\beta}|^2 |\beta H^{-1}|^{2m} \right\}, \quad (3)$$

and also

$$\tilde{L}_2^m(H) = \left\{ f : f(x) = f(x + H\beta), \quad \|f\|_{\tilde{L}_2^m}^2 = \int_{\Omega_0} \sum_{|\alpha|=m} \frac{|\alpha|!}{\alpha!} (D^\alpha f)^2 dx \right\},$$

$\alpha! = \alpha_1! \alpha_2! \dots \alpha_n!$; f_{β} are the Fourier coefficients of the function $f(x)$. Weighted cubature formulas in $\tilde{L}_2^m$ with norm (3) were considered in (7).

Theorem. If $l^{(t)}(x)$ is a functional of the form (1), (2) from $\tilde{L}_2^{m*}$, then

$$\|l^{(t)}\|_{\tilde{L}_2^{m*}}^2 = (2\pi)^{-2m} \sum_{\beta \neq 0} |\beta H^{-1}|^{-2m} \Psi_{H^{-1}}^2(C_{H^{-1}}, \beta), \quad (4)$$

$$\Psi_{H^{-1}}(C_{H^{-1}}, \beta) = \sum_{s=0}^t \sum_{|\alpha|=2s} (-1)^s (2\pi)^{2s} (\beta H^{-1})^\alpha C_{H^{-1}}^{(\alpha)}. \quad (5)$$

Proof. We estimate $(l^{(t)}, \varphi)$ for $\varphi \in \tilde{L}_2^m$, applying the Cauchy inequality:

$$\begin{aligned} |(l^{(t)}, \varphi)| &= \left| (l^{(t)}, \sum_{\beta} \varphi_{\beta} e^{2\pi i(\beta H^{-1}, x)}) \right| \\ &= \left| \sum_{\beta \neq 0} \varphi_{\beta} \left(- \sum_{|\alpha| \leq 2t} (2\pi i)^{|\alpha|} (\beta H^{-1})^\alpha C_{H^{-1}}^{(\alpha)} \right) \right| \\ &= \left| \sum_{\beta \neq 0} (2\pi)^m |\beta H^{-1}|^m \varphi_{\beta} \left(-(2\pi)^{-m} \sum_{|\alpha| \leq 2t} (2\pi i)^{|\alpha|} (\beta H^{-1})^\alpha C_{H^{-1}}^{(\alpha)} |\beta H^{-1}|^{-m} \right) \right| \\ &\leq \left[(2\pi)^{-2m} \sum_{\beta \neq 0} |\beta H^{-1}|^{-2m} \left| \sum_{|\alpha| \leq 2t} (2\pi i)^{|\alpha|} (\beta H^{-1})^\alpha C_{H^{-1}}^{(\alpha)} \right|^2 \right]^{1/2} \|\varphi\|_{\tilde{L}_2^m}. \end{aligned} \quad (6)$$

Put

$$u(x) = -(2\pi)^{-2m} \sum_{\beta \neq 0} \left\{ |\beta H^{-1}|^{-2m} \sum_{|\alpha| \leq 2t} (2\pi i)^{|\alpha|} (\beta H^{-1})^\alpha C_{H^{-1}}^{(\alpha)} \times e^{2\pi i(\beta H^{-1}, x)} \right\};$$

then for $u(x)$ equality is attained in (6):

$$|(l^{(t)}, u)| = \left[(2\pi)^{-2m} \sum_{\beta \neq 0} |\beta H^{-1}|^{-2m} \left| \sum_{|\alpha| \leq 2t} (2\pi i)^{|\alpha|} (\beta H^{-1})^\alpha C_{H^{-1}}^{(\alpha)} \right|^2 \right]^{1/2} \|u\|_{\tilde{L}_2^m}. \quad (7)$$

If in (7) β is replaced by $-\beta$, then the terms with odd $|\alpha|$ cancel each other. The theorem is proved.

Put $y = Hx$ or $x = H^{-1}y$; then

$$\|l^{(t)}\|_{\tilde{L}_2^{m*}}^2 = (2\pi)^{-2m} \sum_{\beta \neq 0} (\beta, \beta^*)^{-m} \Psi^2(c, \beta), \quad (8)$$

$$\Psi(c, \beta) = 1 - \sum_{s=1}^t \sum_{|\alpha|=2s} (-1)^{s-1} (2\pi)^{2s} \beta^\alpha C^{(\alpha)}. \quad (9)$$

Here β^* is conjugate to β , and $\beta^\alpha = \beta_1^{\alpha_1} \beta_2^{\alpha_2} \dots \beta_n^{\alpha_n}$. Put $t \geq n$, $r_0 = [t/n]$, $p = t - nr_0$, $B = \{\beta : |\beta_i| \leq r_0 \ (i = 1, 2, \dots, n)\}$, and solve the following problem.

Problem. Find the coefficients $C^{(\alpha)}$ for (9) so that $\Psi(c, \beta) = 0$ for all integer $\beta \in B$.

The solution of this problem can be represented in the form

$$\Psi(c, \beta) = r_0!^{-2n} (r_0 + 1)^{-2p} \prod_{s=1}^n \prod_{k=1}^{r_0} (\beta_s^2 - k^2) \prod_{s=1}^p [\beta^2 - (r_0 + 1)^2],$$

which grows as $\beta \rightarrow \infty$. Applying the inequality ((9), p. 29), we obtain—we have the following estimate:

$$|\Psi(c, \beta)| \leq r_0!^{-2n} (r_0 + 1)^{-2p} n^{-nr_0} p^{-p} \left(\sum_{s=1}^n \beta_s^2 \right)^t, \quad \beta \notin B. \quad (10)$$

Substituting (10) into (8), we obtain

$$\|l^{(t)}\|_{L_2^{m*}}^2 \leq (2\pi)^{-2m} r_0!^{-4m} (r_0 + 1)^{-4p} n^{-2nr_0} p^{-2p} \sum_{\beta \notin B} (\beta, \beta^*)^{-m+2t}. \quad (11)$$

Let now a periodic lattice be given with period matrix H^{-1} , $|H^{-1}| = 1$. On the basis of (2) (pp. 64–78), we may regard H^{-1} as a triangular matrix.

Put

$$B_{H^{-1}} = \left\{ \beta H^{-1} : \left| \sum_{i=j}^n h_{ij} \beta_i \right| \leq r_1 \ (j = 1, 2, \dots, n) \right\}.$$

Renumber all $\beta H^{-1} \in B_{H^{-1}}$ and, for convenience, introduce the notation

$$\beta^{(k)} H^{-1} = \gamma^{(k)} \in B_{H^{-1}}; \quad a_s = \prod_{k=1}^{\sigma_s} \gamma_s^{(k)}; \quad \gamma_s^{(k)} = \sum_{i=s}^n h_{is} \beta_i^{(k)}; \quad |\gamma_s^{(k)}| \leq r_1;$$

$s = 1, 2, \dots, n; \quad k = 1, 2, \dots, \sigma_s; \quad h_{ij}$ is an element of the matrix H^{-1} , r_1 is chosen

so that the condition $\sum_{k=1}^n \sigma_k = t$ is satisfied.

Solving the preceding problem for this case, we shall have the following estimates for (5) and (4):

$$\Psi_{H^{-1}}(C_{H^{-1}}, \beta) = \prod_{s=1}^n \prod_{k=1}^{\sigma_s} \frac{\gamma_s^2 - \gamma_s^{(k)2}}{a_1^2 a_2^2 \dots a_n^2}; \quad |\Psi_{H^{-1}}| \leq \frac{\sigma_0^{2t} t^{-2t}}{a_1^2 a_2^2 \dots a_n^2} |\beta H^{-1}|^{2t};$$

$$\|l^{(t)}\|_{L_2^{m*}}^2 \leq \frac{(2\pi)^{-2m} \sigma_0^{4t} t^{-4t}}{a_1^4 a_2^4 \dots a_n^4} \sum_{\beta H^{-1} \notin B_{H^{-1}}} |\beta H^{-1}|^{-2m+4t}, \quad \sigma_0 = \max(\sigma_1, \sigma_2, \dots, \sigma_n). \quad (12)$$

We inscribe a ball in B and $B_{H^{-1}}$. The radius of the ball is, obviously, respectively equal to r_0 and r_1 . The estimate

$$\sum_{\beta \notin B} r^{-2m+4t} \leq \frac{\varkappa_n(1+o(1))}{(r_0+1)^{2m-4t-n}}, \quad \varkappa_n = \frac{2\pi^{n/2}}{(2m-4t-n)\Gamma(n/2)}, \quad r^2 = \sum_{s=1}^n \beta_s^2,$$

belongs to L. V. Voĩtishek. Substituting this into (11) and (12), we obtain an estimate of the norm of the functional $l^{(t)}(x)$, respectively for the case of rectangular lattices and lattices with matrix H^{-1} :

$$\|l^{(t)}\|_{L_2^{m*}}^2 \leq \frac{\varkappa_n n^{-2nr_0} p^{-2p} r_0!^{-4n}}{(2\pi)^{2m} (r_0+1)^{2m-4t+4p-n}} + O\left(\frac{1}{(r_0+1)^{2m-4t-n}}\right), \quad (13)$$

$$\|l^{(t)}\|_{L_2^{m*}}^2 \leq \frac{\varkappa_n \sigma_0^{4t} t^{-4t}}{a_1^4 a_2^4 \dots a_n^4 (2\pi)^{2m} (r_1+1)^{2m-4t-n}} + O\left(\frac{1}{(r_1+1)^{2m-4t-n}}\right).$$

Estimate (13) can be transformed into the form

$$\|l^{(t)}(x)\|_{L_2^{m*}}^2 \leq \frac{r_0^{-4n} (r_0+1)^{-2m-4p+4t+n}}{(2m-4t-n)n^{2nr_0}} \|l^{(0)}(x)\|_{L_2^{m*}}^2, \quad (14)$$

where $l^{(0)}(x)$ is obtained from $l^{(t)}(x)$ for $t = 0$.

Example. Put in (14) $n = 2$, $t = 2$, $m = 6$, $r_0 = 1$, $p = 0$; then

$$\|l^{(2)}(x)\|_{L_2^{6*}}^2 \ll \frac{1}{128} \|l^{(0)}(x)\|_{L_2^{m*}}^2.$$

Now let

$$I = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} e^{\sin(x+y)} dx dy.$$

Computations with step $h = \pi/3$ gave the values

$$\|l^{(0)}\|_{\tilde{L}_2^{6*}} = 0.0000450 \dots, \quad \|l^{(2)}\|_{\tilde{L}_2^{6*}} = 0.000000041 \dots$$

Remark. In cases where computing the derivatives of $\varphi(x)$ does not present difficulties, it is advantageous to apply the cubature formula considered in this note.

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Note: Figure translations are in progress. See original paper for figures.

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