

ON THE LIMITING BEHAVIOR OF AN INVARIANT MEASURE UNDER SMALL PERTURBATIONS OF A DYNAMICAL SYSTEM

A. D. Venttsel', M. I. Freidlin

1969

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196901.16186>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

UDC 517.9

MATHEMATICS

A. D. VENTSEL, M. I. FREIDLIN

ON THE LIMITING BEHAVIOR OF AN INVARIANT MEASURE UNDER SMALL PERTURBATIONS OF A DYNAMICAL SYSTEM

(Presented by Academician A. N. Kolmogorov on 21 I 1969)

Let a vector field $\{b^i(x)\} = b(x)$ of class $C^1(S)$ be given on a connected compact manifold S of dimension n . Consider on S the dynamical system

$$\dot{x} = b(x) \quad (1)$$

and the Markov process $X^\varepsilon = \{x_t^\varepsilon, P_x^\varepsilon\}$ (see ^(1,2)), governed by the differential operator

$$L^\varepsilon u(x) = \frac{\varepsilon^2}{2\sqrt{a(x)}} \sum_{i=1}^n \frac{\partial}{\partial x^i} \sum_{j=1}^n a^{ij}(x) \frac{\partial}{\partial x^j} (a(x)u(x)) + \sum_{i=1}^n b^i(x) \frac{\partial u}{\partial x^i},$$

$a^{ij}(x) \in C^1(S)$, $a(x) = \det\{a^{ij}(x)\}^{-1}$, and the quadratic form $\sum_{i,j=1}^n a^{ij}(x)\lambda_i\lambda_j$ is positive definite. Since the manifold S is compact and the operator L^ε is nondegenerate for $\varepsilon > 0$, the process X^ε has a unique stationary distribution (normalized invariant measure) μ^ε . The dynamical system (1), obtained from X^ε for $\varepsilon = 0$, also has a normalized invariant measure, but, generally speaking, not a unique one. It is natural to expect that μ^ε converges as $\varepsilon \rightarrow 0$ to some invariant measure of system (1), but the question arises: to which one exactly? In the present note we study the question of on what set the distribution μ^ε will concentrate as $\varepsilon \rightarrow 0$. In the one-dimensional case, as well as in the case when $b(x)$ is a potential field and $\{a^{ij}(x)\}$ is the identity matrix, an explicit expression for its density may be used to study the limiting behavior of the invariant measure of the process X^ε (see ^(1,3-5)). In the potential case the measure μ^ε as $\varepsilon \rightarrow 0$ concentrates at the point of the absolute minimum of the potential. Theorem 1 below shows what must replace the potential in the general case.

Denote by H the set of continuous, piecewise continuously differentiable functions φ_s , defined on intervals of the real axis of the form $[0, T]$, with values in S . On the elements of H consider the functional

$$I(\varphi) = \int_0^T \sum_{i,j=1}^n a_{ij}(\varphi_s)(\dot{\varphi}_s^i - b^i(\varphi_s))(\dot{\varphi}_s^j - b^j(\varphi_s)) ds,$$

where $\{a_{ij}(x)\}$ is the matrix inverse to $\{a^{ij}(x)\}$, and put

$$V(x, y) = \inf_{\varphi(0)=x, \varphi(T)=y} I(\varphi).$$

The function $V(x, y)$ is continuous in both arguments.

Introduce an equivalence relation for points of the manifold S : $x \sim y$ if $V(x, y) = V(y, x) = 0$. This relation in fact does not depend on the matrix $\{a^{ij}(x)\}$, but is determined only by the structure of the dynamical system (1).

Suppose that the following condition is fulfilled:

Condition A. On the manifold S there exists a finite number of compact sets $K_1, K_2, \dots, K_l$ such that: 1) for any points x, y of one compact set ...

the relation $x \sim y$ holds; 2) if $x \in K_i$, and $y \notin K_i$, then $x \not\sim y$; 3) every α - or ω -limit set (see ⁽⁶⁾) of system (1) belongs entirely to one of the K_i .

Let us note that, when condition 2) is satisfied, a limit set cannot intersect the compact set K_i without belonging to it entirely.

It follows from condition A that a point which does not belong to any of the K_i cannot be equivalent to any other point.

We shall call K_i a **stable set** if $V(x, y) > 0$ for $x \in K_i, y \notin K_i$. The property of stability also depends only on the structure of system (1).

It is easy to see that the function $V(x, y)$ takes one and the same value for all $x \in K_i, y \in K_j$; denote this value by V_{ij} .

We shall call an **i -graph** a graph consisting of $l - 1$ arrows ($j \rightarrow k$), where $j, k = 1, 2, \dots, l, k \neq j$, satisfying the following conditions: 1) j runs through all values from 1 to l , except i ; 2) the graph has no closed cycles.

Denote the set of all i -graphs by G_i ; for its elements we shall use the letter g .

Put

$$V_i = \min_{g \in G_i} \sum_{(j \rightarrow k) \in g} V_{jk}.$$

It is easy to prove that, if $V_i = \min(V_1, V_2, \dots, V_l)$, then the compact set K_i is stable.

Theorem 1. Suppose condition A is satisfied. Denote by M the set of those i , $1 \leq i \leq l$, for which $V_i = \min(V_1, \dots, V_l)$. If B is an arbitrary open set containing $\bigcup_{i \in M} K_i$, then

$$\lim_{\varepsilon \rightarrow 0} \mu^\varepsilon(B) = 1.$$

Theorem 2. Suppose condition A is satisfied, the set M consists of a single element i_0 , and there exists only one normalized invariant measure μ_0 of system (1), concentrated on K_{i_0} . Then the measure μ^ε converges weakly to μ_0 as $\varepsilon \rightarrow 0$.

For the proof of Theorem 1, surround each compact set K_i by a pair of neighborhoods d_i, D_i with smooth boundaries γ_i, Γ_i , respectively, and let $\bar{d}_i \cup \gamma_i \subset D_i$. Denote $\gamma = \bigcup_i \gamma_i$, $\Gamma = \bigcup_i \Gamma_i$. Introduce the random times $\tau_0 = 0$,

$$\sigma_n = \inf\{t \geq \tau_{n-1} : x_t^\varepsilon \in \Gamma\}, \quad \tau_n = \inf\{t \geq \sigma_n : x_t^\varepsilon \in \gamma\}.$$

Consider the Markov chain $z_n = x_{\tau_n}^\varepsilon$ on the phase space γ . It is known ⁽⁷⁾ that the invariant measure μ^ε of the process X^ε can be expressed, up to a factor, in terms of the invariant measure ν^ε of the chain $\{z_n\}$ by the formula

$$\mu^\varepsilon(B) = \int_\gamma \nu^\varepsilon(dy) M_y^\varepsilon \int_0^{\tau_1} \chi_B(x_t^\varepsilon) dt \quad (2)$$

(M_y^ε is the mathematical expectation corresponding to the trajectory x_t^ε issuing from the point y at $t = 0$).

The estimates for the invariant measure ν^ε of the chain $\{z_n\}$ and for the mathematical expectation standing under the integral in (2) which are given below are proved with the aid of Lemmas 1, 2 of ⁽⁸⁾ and the following generalization of Lemma 3 of the same work:

Lemma 1. For any choice of the neighborhoods d_i, D_i there exist constants $T_0, C > 0$ such that, for all sufficiently small ε and for all $x \in \Gamma$, the inequality

$$P_x^\varepsilon\{\tau_1 > T\} \leq \exp[-C(T - T_0)/2\varepsilon^2].$$

The following lemma gives an estimate of the transition probabilities.

Lemma 2. For any $h > 0$ one can indicate an $r > 0$ such that, for any choice of neighborhoods d_i, D_i inside the r -neighborhoods K_i , $i = 1, 2, \dots, l$, there exists $\varepsilon_0 > 0$ such that the probability of transition of the chain $\{z_n\}$ in $s = l - 1$ steps, for $\varepsilon < \varepsilon_0$, $x \in \gamma_i$, satisfies the inequalities

$$\exp\left[\frac{-V_{ij} - h}{2\varepsilon^2}\right] < P^{(s)}(x, \gamma_j) < \exp\left[\frac{-V_{ij} + h}{2\varepsilon^2}\right].$$

For estimating the invariant measure ν^ε , the following lemmas are used.

Lemma 3. Suppose there is a Markov chain with l states for which the transition probabilities $p_{ij} > 0$ for $j \neq i$. Then the stationary distribution of this chain, up to a factor, is $(Q_1, \dots, Q_l)$, where

$$Q_i = \sum_{g \in G_i} \prod_{(j \rightarrow k) \in g} p_{jk}.$$

The same is true if, instead of p_{jk} , one takes the transition probabilities in s steps, $s > 1$.

Lemma 4. Denote by $\underline{Q}_i, \overline{Q}_i$ the quantities obtained by substituting in formula (3), in place of p_{ij} , the lower and upper estimates for the transition probabilities in s steps $p^{(s)}(x, \gamma_i)$, $x \in \gamma_i$ (see Lemma 2). Then the ratio of the values of the invariant measure ν^ε for the sets Γ_i and Γ_j lies between $\underline{Q}_i : \overline{Q}_j$ and $\overline{Q}_i : \underline{Q}_j$.

The mathematical expectations under the integral sign in equality (2) are estimated with the help of the following lemmas.

Lemma 5. If K_i is a stable set, then for any of its neighborhoods D_i one can choose a smaller neighborhood d_i with smooth boundary, $\gamma_i, d_i \cup \gamma_i \subset D_i$, and a constant $a > 0$, such that for all sufficiently small ε

$$M_x^\varepsilon \sigma_1^3 > \exp(a/2\varepsilon^2), \quad x \in \gamma_i.$$

Lemma 6. For any $a > 0$ there exists an $r > 0$ such that, if D_i is contained in the r -neighborhood of K_i , then for all sufficiently small ε

$$M_x^\varepsilon \sigma_1 < \exp(a/2\varepsilon^2), \quad x \in \gamma_i.$$

Moreover, it follows from Lemma 1 that, for ε less than some $\varepsilon_0 > 0$,

$$M_x^\varepsilon \tau_1 \leq T_0 + 2\varepsilon^2/c < T_1 = T_0 + 2\varepsilon_0^2/c, \quad x \in \Gamma.$$

The proof of Theorem 1 is obtained from the lemmas as follows. First choose a positive

$$h < \frac{1}{2l} \left[\min_{i \notin M} V_i - \min_{1 \leq i \leq l} V_i \right].$$

Next choose positive r in accordance with Lemma 2 and fix neighborhoods D_i of those compact sets K_i for which V_i assumes its smallest value ($i \in M$) inside the r -neighborhoods of these compact sets, in such a way that they lie entirely inside the set B . Then, in accordance with Lemma 5, choose their smaller

neighborhoods d_i and a constant $a > 0$; using Lemma 6 with this constant, we find how small the remaining neighborhoods D_i and d_i must be, and fix these neighborhoods. By virtue of the estimates of Lemma 2, the ratios $Q_i : Q_j$ for $i \notin M$, $j \in M$ will tend to zero as $\varepsilon \rightarrow 0$ no more slowly than $\exp(-h/\varepsilon^2)$; therefore, by Lemma 4,

$$\nu^\varepsilon \left(\bigcup_{i \notin M} \gamma_i \right) = O(\exp(-h/\varepsilon^2)), \quad \nu^\varepsilon \left(\bigcup_{i \in M} \gamma_i \right) \rightarrow 1 \quad \text{as } \varepsilon \rightarrow 0.$$

Hence,

$$\begin{aligned} \int_\gamma \nu^\varepsilon(dy) M_y^\varepsilon \int_0^{\tau_1} \chi_B(x_t^\varepsilon) dt &\geq \sum_{i \in M} \int_{\gamma_i} \nu^\varepsilon(dy) M_y^\varepsilon \int_0^{\sigma_1} \chi_B(x_t^\varepsilon) dt \geq \\ &\geq \sum_{i \in M} \int_{\gamma_i} \nu^\varepsilon(dy) M_y^\varepsilon \sigma_1 \geq \text{const} \cdot \exp(a/2\varepsilon^2). \end{aligned}$$

for sufficiently small ε . At the same time

$$\begin{aligned} \int_\gamma \nu^\varepsilon(dy) M_y^\varepsilon \int_0^{\tau_1} \chi_{S \setminus B}(x_t^\varepsilon) dt &= \sum_{i \in M} \int_{\gamma_i} \nu^\varepsilon(dy) M_y^\varepsilon \int_0^{\sigma_1} \chi_{S \setminus B}(x_t^\varepsilon) dt + \\ &+ \sum_{i \notin M} \int_{\gamma_i} \nu^\varepsilon(dy) M_y^\varepsilon \int_0^{\sigma_1} \chi_{S \setminus B}(x_t^\varepsilon) dt + \int_\gamma \nu^\varepsilon(dy) M_y^\varepsilon \int_{\sigma_1}^{\tau_1} \chi_{S \setminus B}(x_t^\varepsilon) dt. \end{aligned}$$

The first integral is equal to zero; the second, for sufficiently small ε , does not exceed

$$\sum_{i \notin M} \int_{\gamma_i} \nu^\varepsilon(dy) M_y^\varepsilon \sigma_1 \leq \text{const} \cdot \exp \left\{ \frac{a - 2h}{2\varepsilon^2} \right\};$$

in the third,

$$M_y^\varepsilon \int_{\sigma_1}^{\tau_1} \chi_{S \setminus B}(x_t^\varepsilon) dt \leq M_y^\varepsilon [\tau_1 - \sigma_1] = M_y^\varepsilon M_{x_{\sigma_1}^\varepsilon}^\varepsilon \tau_1 < T_1.$$

This means that $\mu^\varepsilon(S \setminus B) = O(\mu^\varepsilon(B))$ as $\varepsilon \rightarrow 0$, i.e. $\mu^\varepsilon(B) \rightarrow 1$.

To prove Theorem 2, one verifies that if μ_0 is a weak limit of a subsequence of the measures μ^ε , then the measure μ_0 is invariant.

The authors express their sincere gratitude to A. N. Kolmogorov, who drew their attention to the problem considered in the present work.

Moscow State University
named after M. V. Lomonosov

Received
20 I 1969

REFERENCES

1. A. N. Kolmogoroff, *Math. Ann.*, **113**, 766 (1937).
2. E. B. Dynkin, *Markov Processes*, Moscow, 1963.
3. S. N. Bernstein, *C. R.*, **196**, 1062 (1933).
4. L. S. Pontryagin, A. A. Andronov, A. A. Vitt, *ZhETF*, **3**, 3, 165 (1933).
5. M. B. Nevelson, *Theory of Probability and Its Applications*, **9**, 2, 139 (1964).
6. V. V. Nemytskii, V. V. Stepanov, *Qualitative Theory of Differential Equations*, Moscow–Leningrad, 1949.
7. R. Z. Khasminskii, *Theory of Probability and Its Applications*, **5**, no. 2, 196 (1960).
8. A. D. Venttsel, M. I. Freidlin, *DAN*, **187**, No. 3 (1969).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.