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Abstract

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MATHEMATICS

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ON AN INVARIANT MEASURE FOR A Y -FLOW ON A THREE-DIMENSIONAL MANIFOLD

(Presented by Academician A. N. Kolmogorov, 28 X 1968)

In [1] it is proved that if T is a Y -diffeomorphism of class C^2 of the two-dimensional torus M , having everywhere dense leaves of the transversal foliations, then in M there exists a measure μ , positive on every open set, invariant with respect to T , and T , as an automorphism of the space with measure (M, μ) , is a K -automorphism. Moreover, for every measurable partition ξ of the space M , whose elements C_ξ are intervals of the contracting foliation, the conditional measure $\mu(\cdot | C_\xi)$, induced by μ , is equivalent on almost every C_ξ to the normalized Riemannian volume on C_ξ . The listed properties determine the measure μ uniquely. The proof in [1] is based on the existence of a Markov partition. In the present paper an analogous assertion is proved for a Y -flow on a three-dimensional manifold.

1. Markov partition. Let W be a three-dimensional compact Riemannian manifold of class C^∞ ; $\{T^t\}$ a Y -flow of class C^2 on W (see [2]); Γ_c (Γ_p) its contracting (expanding) foliation; G_c (G_p) the foliation into leaves

$$G_c = \bigcup_{t=-\infty}^{\infty} T^t \Gamma_c \quad \left(G_p = \bigcup_{t=-\infty}^{\infty} T^t \Gamma_p \right).$$

It is assumed that all leaves of the foliations Γ_c and Γ_p are everywhere dense in W .

Let the points w_0 and w_1 lie on one small interval $\gamma_c(w_0)$ of the leaf $\Gamma_c(w_0)$; let M_{w_0} and M_{w_1} be neighborhoods of the points w_0 and w_1 on $G_p(w_0)$ and $G_p(w_1)$ such that for $w \in M_{w_0}$ there is a $w' \in M_{w_1}$, lying on a small interval $\gamma_c(w)$, and the mapping $\pi_{w_0, w_1} : w \rightarrow w'$ is a homeomorphism of M_{w_0} onto M_{w_1} . Let $\gamma_p \in M_{w_0}$ be an interval of the leaf $\Gamma_p(w_0)$; let $\tau(w)$ be a continuous nonnegative function on γ_p , not identically equal to zero.

Every set

$$\Pi = \bigcup_{w \in \gamma_p} \gamma_c(w) \quad \left(V = \bigcup_{w \in \gamma_p} \bigcup_{\tau=0}^{\tau(w)} T^\tau \gamma_c(w) \right)$$

will be called a Γ_c -regular parallelogram (parallelepiped), and the lengths of the intervals γ_p and $\gamma_c(w_0)$ the dimensions of the parallelogram Π . The contracting Δ_c (expanding Δ_p) boundary of V will be called

$$\Delta_c(V) = \bigcup_{y \in \partial \gamma_p} \bigcup_{\tau=0}^{\tau(y)} T^\tau \gamma_c(y) \quad \left(\Delta_p(V) = \bigcup_{w \in \gamma_p} \bigcup_{\tau=0}^{\tau(w)} T^\tau (w \cup \pi_{w_0, w_1}(w)) \right).$$

We shall call Π the upper face of V . If β is a collection of parallelepipeds $\{V_i\}$, then $\Delta_c(\beta) = \bigcup_i \Delta_c(V_i)$ and $\Delta_p(\beta) = \bigcup_i \Delta_p(V_i)$.

Analogously to [1], we introduce the following definition.

Definition. A finite partition α into nonintersecting Γ_c -regular parallelepipeds $\{V_i\}$ is called **Markov** if for all $t \geq 0$, $T^t \Delta_c(\alpha) \subset \Delta_c(\alpha)$, and for all $t \leq 0$, $T^t \Delta_p(\alpha) \subset \Delta_p(\alpha)$.

Theorem 1. For every $\varepsilon > 0$ there exists a Markov partition into Γ_c -regular parallelepipeds whose upper-face dimensions do not exceed ε .

2. Special representation.

Let α be a Markov partition and let M be the set-theoretic union of the upper faces $\{\Pi_i\}$ of the parallelepipeds $\{V_i\}$. Let ξ_c (ξ_p) be the partition of M into intervals C_{ξ_c} (C_{ξ_p}), belonging to $\{\Pi_i \cap \Gamma_c\}$ ($\{\Pi_i \cap C_p\}$). The intervals C_{ξ_p} are not, generally speaking, intervals of leaves Γ_p . However, through each point $y \in C_{\xi}$ one can draw an interval $\gamma C_{\xi_p}(y)$ of the leaf $\Gamma_p(y)$ such that there exists a continuous function $q(z)$, for $z \in \gamma C_{\xi_p}(y)$, having the properties: 1) $q(\gamma C_{\xi_p}(y) \cap C'_{\xi_p}) = 0$; 2) $T^{q(z)}z = C_{\xi_p}$; 3) the mapping $\psi : z \rightarrow T^{q(z)}z$ is a homeomorphism of $\gamma C_{\xi_p}(y)$ onto C_{ξ_p} .

For $x_0 \in \Pi_i$, consider $C_{\xi_c}(x_0)$ and $C_{\xi_p}(x_0)$. Let u_1 (u_2) be a smooth coordinate on the leaf $\Gamma_c(x_0)$ ($\Gamma_p(x_0)$) such that $u_1(x_0) = 0$ ($u_2(x_0) = 0$). Consider on $C_{\xi_c}(x_0)$ the coordinate u_1 , and on $C_{\xi_p}(x_0)$ the coordinate u_2 , transferred from $\gamma C_{\xi_p}(x_0)$ by means of the mapping ψ . For every point $w \in \Pi$ there are points $z_1 \in C_{\xi_c}(x_0)$ with coordinate u_1 and $z_2 \in C_{\xi_p}(x_0)$ with coordinate u_2 such that $w = C_{\xi_c}(z_2) \cap C_{\xi_p}(z_1)$. Assign to w the coordinates (u_1, u_2) . Then Π_i becomes a smooth manifold with boundary, and M will consist of a finite number of such manifolds, generally not connected with one another. Consider on Π_i the mapping $\chi : w \rightarrow (u_1, u_2)$. It is a homeomorphism of Π_i onto a rectangle P_i of the Euclidean plane (u_1, u_2) . We introduce on Π_i Riemannian lengths transferred from P_i by means of the mapping χ^{-1} .

From the absolute continuity of the foliations G_p and G_c (see (2)) it follows that the Riemannian length on C_{ξ_c} in Π_i is equivalent to the Riemannian length on C_{ξ_c} induced by the Riemannian metric in W , while the Riemannian length on C_{ξ_p} is equivalent to the Riemannian length on C_{ξ_p} , transferred from $\gamma C_{\xi_p}(x)$, $x \in C_{\xi_p}$, by means of the mapping ψ .

Through $x \in M$ draw an interval of length $l(x)$ of the trajectory of the flow $\{T^t\}$ in the positive direction up to the first intersection with M at the point x' . The mapping $U : x \rightarrow x'$ is a one-to-one mapping of M onto itself, and the flow $\{T^t\}$ is represented as a special flow (see (3)) over (M, U) with function $l(x)$. The mapping U on M is similar in its properties to a Y -diffeomorphism: it uniformly exponentially contracts (expands) the intervals $C_{\xi_c}(C_{\xi_p})$. The mapping U cannot be called a Y -diffeomorphism, since M is not a connected closed manifold and U has, generally speaking, discontinuity points.

The partition $\{\Pi_i\}$ is Markov for U , and the partition ξ_c is increasing. The function $l(x)$ is constant on C_{ξ_c} . Using the Hölder condition for the foliation Γ_c (see (4)), one can prove that $l(y)$, $y \in C_{\xi_p}$, satisfies a Hölder condition of positive order on each component of its continuity.

Using (4) and the absolute continuity of the foliation G_p , by the methods of the work [1] one can prove the following theorem.

Theorem 2. *In the space M there exists a measure m , positive on every open set, invariant with respect to U , and U , as an automorphism of the space M with measure m , is a K -automorphism with K -partition ξ_c . The conditional measure $m(\cdot | C_{\xi_c})$, induced by m on almost every C_{ξ_c} , is equivalent to the normalized Riemannian volume on C_{ξ_c} . These properties determine the measure m uniquely.*

3. Invariant measure.

Using the special representation of the flow $\{T^t\}$, consider in W the normalized measure

$$d\mu = \frac{1}{\bar{l}}(dm \times dt).$$

$$\bar{l} = \int_M l(x) dm$$

and the partition η_c , consisting of the segments

$$\{T^u C_{\xi_c}(x), 0 \leq u < l(x), x \in M\}.$$

From the invariance of the measure m with respect to U , the invariance of the measure μ with respect to the flow $\{T^t\}$ follows easily. The partition η_c has the following properties: 1) $T^{t_1} \eta_c < T^{t_2} \eta_c$ for $t_1 < t_2$;

2)

$$\prod_{t=-\infty}^{\infty} T^t \eta_c = \varepsilon,$$

where ε is the partition into individual points; 3)

$$\bigcap_{t=-\infty}^{\infty} T^t \eta_c = \nu_Z,$$

where ν_Z is the measurable envelope of the partition Z into full layers of the foliation Γ_c .

It follows from (5) that if the foliations Γ_c and Γ_p form a nonintegrable pair, then $\nu_Z = \nu$, where ν is the trivial partition whose only element is the whole space.

Thus the following theorem has been proved:

Theorem 3. In the space W there exists a measure μ , positive on every open set, invariant with respect to the flow $\{T^t\}$, and such that the conditional measure $\mu(\cdot/C_{\eta_c})$, induced by μ on almost every C_{η_c} , is equivalent to the normalized Riemannian volume on C_{η_c} . If the foliations Γ_c and Γ_p form a nonintegrable pair in the sense of (5), then the flow $\{T^t\}$ in (W, μ) is a K -flow with K -partition η_c . The measure μ is uniquely determined by these properties.

For the geodesic flow on a compact manifold of negative curvature, the measure μ coincides with the invariant Riemannian volume.

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