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Abstract

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MATHEMATICS

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ON THE THEORY OF SINGULAR INTEGRAL EQUATIONS WITH MEASURABLE COEFFICIENTS

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Let a simple closed oriented Lyapunov contour Γ divide the complex plane into an exterior domain D^- and an interior domain D^+ , and let the point $z = 0$ belong to D^+ . By Λ we denote the set of functions piecewise continuous on Γ , and by $\mathcal{L}$ the closure of this set in the sense of uniform convergence. Following ⁽¹⁾, we shall call functions in $\mathcal{L}$ lineatchatye.

In the present note we investigate the problem of perturbations of a singular operator with bounded measurable coefficients and establish a general theorem on the solvability of singular integral equations in the space L_p with coefficients from $\mathcal{L}$.

Consider the singular integral equation in the space $L_p(\Gamma)$

$$d(t)(S\varphi)(t) + c(t)\varphi(t) = g(t), \quad (1)$$

where $c(t)$ and $d(t)$ are measurable and essentially bounded functions on Γ , and

$$(S\varphi)(t) = \frac{1}{\pi i} \int_{\Gamma} \frac{\varphi(\tau)}{\tau - t} d\tau \quad (t \in \Gamma).$$

With the aid of the projectors $P = (I + S)/2$ and $Q = (I - S)/2$, the integral equation (1) is rewritten in the following form:

$$aP + bQ = g,$$

where $a(t) = c(t) + d(t)$, $b(t) = c(t) - d(t)$.

Theorem 1. *Let $a(t)$ be a bounded measurable function and let the operator $A = aP + Q$ be a Φ -operator in the space $L_p(\Gamma)$. Then there exists a number r ($1 < r < \infty$) such that, for any function $c(t)$ of the class $A[r]$, the operator $B = acP + Q$ is a Φ -operator, and*

$$\dim \ker B = \max(0, \operatorname{ind} A - \operatorname{ind} c), \quad \dim \operatorname{coker} B = \max(0, \operatorname{ind} c - \operatorname{ind} A).$$

Proof. For the Φ -operator A there exists a number $\eta > 0$ ⁽³⁾ such that any operator A_0 for which $\|A_0 - A\| < \eta$ is a Φ -operator and $\text{ind } A_0 = \text{ind } A$. Choose a number $r(> 1)$ from the condition

$$\max[r, r'] > \pi \sup |a| \|P\| / \eta,$$

where $1/r + 1/r' = 1$.

Let now $c(t)$ be any function of the class $A[r]$; then ⁽²⁾

$$c(t) = c_1(t)c_2(t),$$

where $c_1(t)$ is a Hölder function and $\text{ind } c_1 = \text{ind } c = \chi$, while $c_2(t)$ is a function of the class $A[r]$ and

$$|\arg c_2| < \pi / \max[r, r'] - \varepsilon.$$

Since for the function

$$m(t) = c_2(t)/|c_2(t)| - 1$$

we have

$$|m(t)| < \sup_{t \in \Gamma} |\arg c_2(t)| < \eta / \sup |a| \|P\|,$$

it follows that

$$K = a(1 + m)P + Q$$

is a Φ -operator and $\text{ind } K = \text{ind } A$.

Let $\chi \geq 0$. Using the factorization of the functions $c_1 = f_+ t^\chi f_-$ and $|c_2| = g_+ g_-$, where the functions $f_+ f_-$, g_+ , g_- and their inverses are bounded on the contour Γ (see ⁽²⁾), the operator B can be represented in the form

$$B = X(a(1 + m)P + Q)Y(t^\chi P + Q),$$

where

$$X = g_- f_- I$$

and

$$Y = g_+ f_+ P + g_-^{-1} f_-^{-1} Q$$

are invertible

* For the definition of the class $A[r]$ and the index of a function $c(t) \in A[r]$, see ⁽²⁾.

operators. Taking into account that the operator $t^\chi P + Q$ is a Φ -operator and $\text{ind}(t^\chi P + Q) = -\chi$, we obtain that B is a Φ -operator, $\text{ind } B = \text{ind } A - \chi = \text{ind } A - \text{ind } c$, and by virtue of ^(4, 5) it is left invertible.

For $\chi < 0$ the equality

$$B(t^{-\chi} P + Q) = X(a(1 + m)P + Q)Y$$

holds, whence it follows that the operator B is right invertible and $\text{ind } B = \text{ind } A - \chi$. The theorem is proved.

Corollary. If $A = aP + Q$ is a Φ -operator, then there exists a number $l(a, p)$ (> 0) such that, for every measurable function $c(t)$ satisfying the conditions $0 < N \leq |c(t)| \leq M < \infty$ and

$$\sup_{t \in \Gamma} |\arg c(t)| < l(a, p),$$

the operator $B = acP + Q$ will be a Φ -operator and $\text{ind } B = \text{ind } A$.

For the case of a piecewise continuous function $a(t)$, one can indicate the dependence of the quantity $l(a, p)$ on the function $a(t)$ and the space L_p . First let us note some properties of functions from Λ and $\mathcal{L}$.

Following ⁽⁶⁾, to each function $a(t) \in \mathcal{L}$ we associate, in the natural way, the curve with orientation

$$V_p(a)(\mu, t) = a(t-0)M(p, \mu) + a(t+0)(1 - M(p, \mu))$$

$$(0 \leq \mu \leq 1, t \in \Gamma),$$

where

$$M(p, \mu) = \exp\{(1 - \mu)\theta\} \sin \mu\theta / \sin \theta, \quad \theta = \pi(1 - 2/p),$$

if $p \neq 2$, and $V_2(a) = a(t-0)\mu + a(t+0)(1 - \mu)$. A function $a(t)$ from $\mathcal{L}$ is called p -nonsingular if

$$\inf_{t, \mu} |V_p(a)| > 0.$$

Let $a(t)$ be a p -nonsingular function from $\mathcal{L}$, t_0 an arbitrary point of continuity, and $t_1, t_2, \dots$ its discontinuity points; then the ratio $a(t_k - 0)/a(t_k + 0)$ can be represented in the form

$$|a(t_k - 0)/a(t_k + 0)| \exp\{ih_k(a)\} \quad (k = 0, 1, 2, \dots),$$

where

$$-2\pi/q < h_k(a) < 2\pi/p, \quad 1/p + 1/q = 1.$$

To each such function one can assign two numbers $h_+(a)$ and $h_-(a)$, defining them by the equalities

$$h_+(a) = \max_k h_k(a), \quad h_-(a) = \min_k h_k(a).$$

Obviously,

$$-2\pi/q < h_-(a) \leq h_k(a) \leq h_+(a) < 2\pi/p. \quad (2)$$

For p -nonsingular functions from $\mathcal{L}$ the index is introduced in the following way. Let a sequence of functions $a_n(t)$ from Λ converge uniformly to $a(t)$. Then, starting from some n , the functions $a_n(t)$ are p -nonsingular and their p -index is

equal to one and the same number; this number is taken to be the p -index of the function $a(t)$ from $\mathcal{L}$.

Theorem 2. For a p -nonsingular function $a(t)$ from Λ , as the number $l(a, p)$ one may take

$$\min[2\pi - ph_+(a), 2\pi + qh_-(a)]/2 \max[p, q] \quad (1/p + 1/q = 1).$$

Moreover, if $\chi (= \text{ind}_p a) = 0$, the operator $A = acP + Q$ is invertible in the space $L_p(\Gamma)$; if $\chi > 0$, the operator A is left invertible and $\dim \text{coker } A = \chi$; if $\chi < 0$, the operator A is right invertible and $\dim \ker A = |\chi|$.

Remark 1. The number $l(a, p)$ indicated in the theorem is, in a certain sense, sharp. Indeed, take on the unit circle Γ_0 the piecewise continuous functions

$$c = a = \exp\{i\theta/4\} \quad (-\pi < \theta \leq \pi);$$

then

$$\max_{\epsilon \in \Gamma_0} |\arg c| = l(a, p),$$

and the operator $caP + Q$ in the space $L_2(\Gamma_0)$ is not invertible from either side (see (6)).

Proof of Theorem 2. From the discontinuity points t_k of the function $a(t)$ and the numbers $h_k(a)$ we construct the function

$$\psi(t) = \prod_{k=1}^m t^{\gamma_k}, \quad \text{where } \gamma_k = \frac{h_k(a)}{2\pi} + \frac{1}{2\pi i} \ln \left| \frac{a(t_k - 0)}{a(t_k + 0)} \right|,$$

and the function t^{γ_k} has a discontinuity only at the point t_k ($k = 1, 2, \dots, m$). Then the function $b(t) = a(t)\psi^{-1}(t)$ is continuous on the contour Γ , and $\text{ind } b = \text{ind}_p a$. Using the condition of the theorem, choose a number $\varepsilon > 0$ so that

$$|\arg c| < (\min[2\pi - ph_+(Q), 2\pi + qh_-(Q)] - \varepsilon)/2 \max[p, q].$$

Let $r(t)$ be such—

a rational function such that $b(t) = r(t)(1 + m(t))$, $\text{ind } r(t) = \text{ind } b(t)$, $|\arg(1 + m)| < \varepsilon/8 \max[p, q]$. The functions r, ψ , and $c_1 = c(1 + m)$ admit the factorization (2, 7)

$$r = r_+ t^X r_-, \quad \psi = \psi_+ \psi_-, \quad c(1 + m) = c_+ c_-.$$

The factorization factors for the functions ψ and c_1 can be represented in the form

$$\psi_{\pm} = F_{\pm} \rho_{\pm}^{\pm 1}, \quad \text{where } \sup_t |F_{\pm}^{\pm 1}(t)| < \infty, \quad \rho_1(t) = \prod_{k=1}^m |t - t_k|^{\text{Re } \gamma_k},$$

$$c_{\pm} = u_{\pm} \rho_2^{\pm 1}, \quad \text{where } \sup_t |u_{\pm}^{\pm 1}(t)| < \infty, \quad \rho_2(t) = \exp \left\{ \frac{1}{2\pi} \int_{\Gamma} \frac{\arg c_1(\tau)}{\tau - t} d\tau \right\}.$$

Consider the operator $Z = (\rho_1 \rho_2)^{-1} S \rho_1 \rho_2$; its boundedness in the space $L_p(\Gamma)$ is proved by applying Stein's theorem⁽⁸⁾. Indeed, take $\gamma = 2\pi / (\min[2\pi - ph_+(a), 2\pi + qh_-(a)] - \varepsilon/2) (> 1)$; then $|\gamma \arg c_1(t)| < \pi / \max[p, q] - \varepsilon_1$ ($\varepsilon_1 > 0$), and by I. B. Simonenko's theorem on weights⁽²⁾

$$\|(Sf)\rho_2^{-\gamma}\|_{L_p} \leq M_1 \|f\rho_2^{-\gamma}\|_{L_p}. \quad (3)$$

Moreover, since the function $a(t)$ is p -nonsingular, for the chosen γ the relations

$$-1/q < \operatorname{Re} \gamma_k \cdot \gamma / (\gamma - 1) < 1/p \quad (k = 1, 2, \dots, m)$$

follow from (2).

Then from B. V. Khvedelidze's theorem on the boundedness of singular operators in L_p spaces with a weight⁽⁷⁾, p. 24) it follows that

$$\|(Sf)\rho_1^{-\gamma/(\gamma-1)}\|_{L_p} \leq M_2 \|f\rho_1^{-\gamma/(\gamma-1)}\|_{L_p}. \quad (4)$$

Since for $t = 1/\gamma$

$$\rho_2^{-\gamma t} \rho_1^{-(1-t)\gamma/(\gamma-1)} = (\rho_2 \rho_1)^{-1},$$

from (3) and (4), by virtue of Stein's interpolation theory, the boundedness of the operator Z follows.

Let $\operatorname{ind}_p a = \chi = 0$; then the operator A can be represented in the form

$$A = acP + Q = r_- \psi_- c_- (r_+ \psi_+ c_+ P + r_-^{-1} \psi_-^{-1} c_-^{-1} Q).$$

Consider the operator

$$B = (r_+^{-1} \psi_+^{-1} c_+^{-1} P + r_- \psi_- c_- Q) r_-^{-1} \psi_-^{-1} c_-^{-1}.$$

Its boundedness follows easily from the boundedness of the operator Z . It is not difficult to verify that the operator B is inverse to A . Thus, for $\chi = 0$ the operator $acP + Q$ is invertible in $L_p(\Gamma)$.

Let $\chi > 0$; then $\operatorname{ind}(at^{-\chi}) = 0$, and from what was proved above it follows that the operator $t^{-\chi}acP + Q$ is invertible in $L_p(\Gamma)$. Since the operator $t^{\chi}P + Q$ is left invertible, it follows from the equality

$$(act^{-\chi}P + Q)(t^{\chi}P + Q) = acP + Q$$

that the operator A is left invertible and $\dim \operatorname{coker} A = \chi$.

If $\chi < 0$, then

$$act^{-\chi}P + Q = (acP + Q)(t^{-\chi}P + Q),$$

and, consequently, the operator A is right invertible and $\dim \ker A = |\chi|$.

Remark 2. The proof given above extends without difficulty to the case of a p -nonsingular function $a(t)$ from $\mathcal{L}$.

Theorem 3. Let $a(t), b(t) \in \mathcal{L}$. In order that the operator

$$\hat{A} = aP + bQ$$

be a $\Phi_+(\Phi_-)$ -operator in the space $L_p(\Gamma)$, it is necessary and sufficient that: 1) $\operatorname{ess\,inf}_{t \in \Gamma} |b(t)| > 0$; 2) the function $a(t)/b(t)$ be p -nonsingular.

If these two conditions are fulfilled and $\chi = \operatorname{ind}_p(a/b)$, then for $\chi = 0$ the operator $\hat{A}$ is invertible in the space $L_p(\Gamma)$; for $\chi > 0$ the operator $\hat{A}$ is left invertible and $\chi = \dim \operatorname{coker} \hat{A}$; for $\chi < 0$ the operator $\hat{A}$ is right invertible and $|\chi| = \dim \ker \hat{A}$.

Proof. The sufficiency of the conditions of the theorem follows from Remark 2. Necessity is established as follows. Let $\hat{A} = aP + bQ$ be a $\Phi_+(\Phi_-)$ -operator and

$$\operatorname{ess\,inf}_{t \in \Gamma} |b(t)| = 0.$$

Since the set of $\Phi_+(\Phi_-)$ -operators is open, one can choose functions $\tilde{a}$ and $\tilde{b}$ from Λ such that

$$\operatorname{ess\,inf}_{t \in \Gamma} |\tilde{b}(t)| = 0$$

and $\tilde{a}P + \tilde{b}Q$ is a $\Phi_+(\Phi_-)$ -operator, which leads to a contradiction with (6). Thus

$$\operatorname{ess\,inf}_{t \in \Gamma} |b(t)| > 0.$$

Now the problem reduces to the study of the operator

$$A_0 = (a/b)P + Q.$$

Let A_0 be a $\Phi_+(\Phi_-)$ -operator and let a/b not be a p -nonsingular function. Then there exists a function $g(t) \in \Lambda$ such that

$$\inf_{t \in \Gamma, 0 \leq \mu \leq 1} |V_p(g)(\mu, t)| = 0$$

and the operator $gP + Q$ is a $\Phi_+(\Phi_-)$ -operator. This leads to a contradiction. The theorem is proved.

Let us note that the sufficiency of the conditions of Theorem 3 for the case when

$$\sum |h_k(a)| < \infty$$

is established in ⁽⁹⁾.

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Note: Figure translations are in progress. See original paper for figures.

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