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Abstract

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MATHEMATICS

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ON SOME PROBLEMS OF NONLINEAR STOCHASTIC PROGRAMMING

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Problems of planning and control under conditions of incomplete information can be written in terms of stochastic programming (see, for example, ⁽¹⁾). Published works on stochastic programming concern mainly linear problems. In ⁽²⁾ methods are studied for the numerical solution of convex stochastic conditional extremum problems. In the present paper approaches are considered to the formulation and solution of a fairly broad class of nonlinear (including nonconvex) stochastic problems. A number of known models of stochastic programming fit into the proposed scheme.

1°. Let (Ω, F, p) be a probability space on the compact set Ω , and let X, Y be separable B -spaces. To each $\omega \in \Omega$ there are assigned nonempty sets $G_0(\omega), \dots, G_n(\omega) \subset X$ and a linear operator $A(\omega) : X \rightarrow Y$ defined on the whole space X .

It is assumed that: a) all the sets $\{x, \omega : x \in G_i(\omega)\}$, $i = 0, 1, \dots, n$, are Borel; b) $G_0(\omega) \neq X$ with probability 1; c) $A(\omega)$ is a measurable operator-function, i.e., for any measurable $x(\omega)$ the function $y(\omega) = A(\omega)x(\omega)$ is measurable.

Problem I. It is required to minimize

$$P\{x(\omega) \in G_0(\omega)\} \quad (1)$$

under the conditions

$$P\{x(\omega) \in G_i(\omega)\} \geq \alpha_i, \quad i = 1, \dots, n; \quad (2)$$

$$M\{A(\omega)x(\omega)\} \leq b, \quad b \in Y. \quad (3)$$

In what follows the case $n = 2$ is considered. The same methods are applicable also in the general case. The additional difficulties arising here are of a purely combinatorial nature.

2°. Consider the auxiliary Problem II, obtained from Problem I by discarding condition (3).

Problem II. It is required to minimize

$$P\{x(\omega) \in G_0(\omega)\}$$

under the conditions

$$P\{x(\omega) \in G_0(\omega)\} \geq \alpha_i, \quad i = 1, 2.$$

The events corresponding to the following nine situations are pairwise incompatible and the probability of at least one of them is equal to 1.

- (1) $(G_1 \cap G_2) \setminus G_0 \neq \emptyset$;
- (2) $\emptyset \neq G_1 \cap G_2 \subset G_0$, $G_1 \setminus G_0 \neq \emptyset$, $G_2 \setminus G_0 \neq \emptyset$;
- (3) $\emptyset \neq G_1 \cap G_2 \subset G_0$, $G_1 \setminus G_0 \neq \emptyset$, $G_2 \setminus G_0 = \emptyset$;
- (4) $\emptyset \neq G_1 \cap G_2 \subset G_0$, $G_1 \setminus G_0 = \emptyset$, $G_2 \setminus G_0 \neq \emptyset$;
- (5) $\emptyset \neq G_1 \cap G_2 \subset G_0$, $G_1 \setminus G_0 = \emptyset$, $G_2 \setminus G_0 = \emptyset$;
- (6) $G_1 \cap G_2 = \emptyset$, $G_1 \setminus G_0 \neq \emptyset$, $G_2 \setminus G_0 \neq \emptyset$;
- (7) $G_1 \cap G_2 = \emptyset$, $G_1 \setminus G_0 \neq \emptyset$, $G_2 \setminus G_0 = \emptyset$;
- (8) $G_1 \cap G_2 = \emptyset$, $G_1 \setminus G_0 = \emptyset$, $G_2 \setminus G_0 \neq \emptyset$;
- (9) $G_1 \cap G_2 = \emptyset$, $G_1 \setminus G_0 = \emptyset$, $G_2 \setminus G_0 = \emptyset$.

Denote by p_i the probability of event (i).

Obviously, for the consistency of problem II it is necessary and sufficient that

$$0 \leq \alpha_i \leq 1 \quad (i = 1, 2), \quad \alpha_1 + \alpha_2 + p_6 + p_7 + p_8 + p_9 \leq 2. \quad (4)$$

In what follows condition (4) is assumed to be satisfied.

Introduce the functions

$$\varphi_i(x, \omega) = \begin{cases} 1, & x \in G_i(\omega), \\ 0, & x \notin G_i(\omega). \end{cases}$$

Then

$$P\{x(\omega) \in G_i(\omega)\} = \int_{\Omega} \varphi_i(\omega, x(\omega)) dp_{\omega}.$$

Problem II is transformed into an extremal problem with an integral functional and integral constraints, and the methods developed in (3) are applicable to its solution.

Denote by $S_2(\alpha_1, \alpha_2)$ the required lower bound of the functional (1) under the conditions (2).

Theorem 1. $S_2(\alpha_1, \alpha_2)$ is equal to the greatest of the following 7 numbers

$$\begin{aligned} \text{(I)} \quad & \alpha_1 + \alpha_2 - 1 - (p_1 - p_9), \\ \text{(II)} \quad & \alpha_2 - p_1 - p_2 - p_4 - p_6 - p_8, \\ \text{(III)} \quad & 2\alpha_1 + \alpha_2 - 2 - (p_1 - p_8 - p_9), \\ \text{(IV)} \quad & \alpha_1 - p_1 - p_2 - p_3 - p_6 - p_7, \\ \text{(V)} \quad & \alpha_1 + 2\alpha_2 - 2 - (p_1 - p_7 - p_9), \\ \text{(VI)} \quad & \frac{1}{2}[\alpha_1 + \alpha_2 - 1 - (p_1 - p_5 - p_9)], \\ \text{(VII)} \quad & 0. \end{aligned} \tag{5}$$

We shall now clarify the nature of the solution of the problem. Consider 7 equations

$$\begin{aligned}
 \text{(I)} \quad \varphi_1 + \varphi_2 - \varphi_0 &= \begin{cases} 2 & \text{in situation (1),} \\ 1 & \text{in (2) } \div (8), \\ 0 & \text{in (9);} \end{cases} \\
 \text{(II)} \quad \varphi_2 - \varphi_0 &= \begin{cases} 1 & \text{in situations (1), (2), (4), (6), (8),} \\ 0 & \text{in (3), (5), (7), (9);} \end{cases} \\
 \text{(III)} \quad 2\varphi_1 + \varphi_2 - \varphi_0 &= \begin{cases} 3 & \text{in situation (1),} \\ 2 & \text{in (2) } - (7), \\ 1 & \text{in (8), (9);} \end{cases} \\
 \text{(IV)} \quad \varphi_1 - \varphi_0 &= \begin{cases} 1 & \text{in situations (1) } \div (3), (6), (7), \\ 0 & \text{in (4), (5), (8), (9);} \end{cases} \quad (6) \\
 \text{(V)} \quad \varphi_1 + 2\varphi_2 - \varphi_0 &= \begin{cases} 3 & \text{in situation (1),} \\ 2 & \text{in (2) } - (6), (8), \\ 1 & \text{in (7), (9);} \end{cases} \\
 \text{(VI)} \quad \frac{\varphi_1 + \varphi_2}{2} - \varphi_0 &= \begin{cases} 1 & \text{in situation (1),} \\ \frac{1}{2} & \text{in (2) } - (4), (6) - (8), \\ 0 & \text{in (5), (9);} \end{cases} \\
 \text{(VII)} \quad \varphi_0 &= 0.
 \end{aligned}$$

Theorem 2. Suppose $S_2(\alpha_1, \alpha_2)$ is equal to the i -th number of system (5). Then there exists a solution $x(\omega)$ of problem II satisfying conditions (2) and the i -th equation of system (6).

3°. Let us return to problem I. We shall call a sequence $x_n(\omega)$ nearly admissible if

$$\lim_{n \rightarrow \infty} P\{x_n(\omega) \in G_i(\omega)\} \geq \alpha_i, \quad M(A(\omega)x_n(\omega)) = y'_n + y''_n,$$

where $y'_n \leq b$, $y''_n \rightarrow 0$.

If the set of almost admissible sequences is nonempty, then problem I is called weakly consistent. With each almost admissible sequence $\{x_n(\omega)\}$ we associate the functional

$$\lim_{n \rightarrow \infty} P\{x_n(\omega) \in G_0(\omega)\}$$

and denote by $S(a_1, a_2)$ its greatest lower bound over all almost admissible sequences, and by $S_1(a_1, a_2)$ the greatest lower bound of (1) in problem I.

Let L be the set of elements of the space Y that can be represented in the form

$$y = M\{A(\omega)x(\omega)\} = \int_{\Omega} A(\omega)x(\omega) dp_{\omega}.$$

Obviously, L is a linear manifold.

Theorem 3. Problem I is weakly consistent if and only if problem II is consistent and there exists an element $y \in \bar{L}$ (the closure of L) satisfying the inequality $y \leq b$.

If, moreover, Y is finite-dimensional, then weak consistency of problem I is equivalent to its consistency.

Theorem 4. If problem I is weakly consistent, then

$$S(a_1, a_2) = S_2(a_1, a_2).$$

If, moreover, Y is finite-dimensional, then

$$S_1(a_1, a_2) = S_2(a_1, a_2).$$

Let $F \subset \Omega$ be an arbitrary measurable set, and let L_F be the collection of elements $y \in Y$ represented in the form

$$y = \int_F A(\omega)x(\omega) dp_{\omega}.$$

For any $\omega \in \Omega$, denote by $L_{\omega} = \bigcap L_F$, where the intersection is taken over all neighborhoods F of the point ω . Finally, let $\sum_{\omega \in \Omega} L_{\omega}$ be the collection of all possible finite sums $y_1 + \dots + y_n$, $y_i \in L_{\omega_i}$.

Theorem 5. Let problem I be weakly consistent. Then for any $\varepsilon > 0$ there exist a function $x(\omega)$ and a vector $y \in \sum L_{\omega}$, $y = y_1 + \dots + y_n$, $y_i \in L_{\omega_i}$, such that:

- a) $P\{x(\omega) \in G_0(\omega)\} \leq S_1(a_1, a_2) + \varepsilon$;
- b) $P\{x(\omega) \in G_i(\omega)\} \geq a_i - \varepsilon$, $i = 1, 2$;
- c) $M\{A(\omega)x(\omega)\} + y = y' + y''$, where $y' \leq b$, $\|y''\| < \varepsilon$;
- d) $x(\omega)$, with probability $1 - \varepsilon$, satisfies Theorem 2.

Corollary. Let $Y = R^n$ and let problem I be consistent. Then, if $x_0(\omega)$ is a solution of problem II, there exist points $\omega_1, \dots, \omega_r$ ($r \leq n$) and vectors $y_1, \dots, y_r$, $y_i \in L_{\omega_i}$, such that

$$M\{A(\omega)x_0(\omega)\} + \sum y_i \leq b.$$

From the last corollary and the definition of L_ω , the structure of almost admissible minimizing sequences for problem I follows at once. These sequences are formed by functions $x_n(\omega)$ representable as the sum

$$x_n(\omega) = x_0(\omega) + \sum_i x_{in}(\omega),$$

where $x_{in}(\omega)$ is different from zero only in some neighborhood of the point ω_i , having measure not exceeding $1/n$, and

$$\int_{\Omega} A(\omega)x_{in}(\omega) dp_{\omega} = y_i, \quad i = 1, 2, \dots, r \leq n.$$

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References

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2. D. B. Yudin, *Economics and Mathematical Methods*, No. 6 (1968).
3. A. D. Ioffe, V. M. Tikhomirov, *Doklady Akademii Nauk*, 184, No. 4 (1968).

Note: Figure translations are in progress. See original paper for figures.

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