

SYMMETRY AND NONLINEAR TENSOR FUNCTIONS IN MINKOWSKI SPACE

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Abstract

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MATHEMATICS

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SYMMETRY AND NONLINEAR TENSOR FUNCTIONS IN MINKOWSKI SPACE

(Presented by Academician L. I. Sedov on 15 X 1968)

1. We shall call ⁽¹⁾ a tensor T a tensor function of the tensors $H_{(1)}, \dots, H_{(s)}$ if the components of the tensor T , independently of the choice of coordinate system, are the same functions of the components of the tensors $H_{(1)}, \dots, H_{(s)}$.

By the symmetry group $\mathcal{G}_H$ of a tensor H we shall mean the totality of all coordinate transformations that do not change the values of the components of the tensor H (in any coordinate system). We shall also say that the tensor H determines the group $\mathcal{G}_H$ and that the tensor H is invariant with respect to a group $\mathcal{G}$, if $\mathcal{G} \subseteq \mathcal{G}_H$.

The symmetry group $\mathcal{G}_T$ of the tensor function T contains the intersection of the symmetry groups of the tensor arguments $H_{(1)}, \dots, H_{(s)}$.

The structure of the tensor function T is determined by its symmetry group $\mathcal{G}_T$. Namely, the set of tensors of a given rank invariant with respect to the group $\mathcal{G}_T$, obviously, forms a finite-dimensional linear space. If $T_{(1)}, \dots, T_{(p)}$ is a basis of this space, then

$$T = k_1 T_{(1)} + \dots + k_p T_{(p)},$$

where $k_1, \dots, k_p$ are certain scalars depending on the tensors $H_{(1)}, \dots, H_{(s)}$.

Thus, the problem of the structure of the tensor function T is reduced to determining the symmetry group $\mathcal{G}_T$ and constructing a tensor basis $T_{(1)}, \dots, T_{(p)}$ for a given set of arguments $H_{(1)}, \dots, H_{(s)}$.

2. The solution of the problem is based on the following easily proved theorems.

Theorem 1. *The symmetry group of any tensor is either a finite group or a continuous Lie group.*

Let $\mathcal{G}_H$ be the symmetry group of the set of tensors $H_{(1)}, \dots, H_{(s)}$, in other words, the tensors $H_{(1)}, \dots, H_{(s)}$ determine the group $\mathcal{G}_H$.

Theorem 2. Any tensor T of arbitrary rank, invariant with respect to the group $\mathcal{G}_H$, can be represented in the form of a linear combination of tensors composed, by means of the operations of tensor multiplication and contraction, from the tensors $H_{(1)}, \dots, H_{(s)}$.

In physical applications, among the arguments $H_{(1)}, \dots, H_{(s)}$ of the tensor function T there is usually contained the metric tensor g , which determines the Lorentz group $\mathcal{L}$. It is therefore of interest to find all subgroups of the Lorentz group that are symmetry groups of tensors, and in effect to construct sets of tensors that determine these subgroups.

Theorem 3. Every finite subgroup $\mathcal{K}$ of the Lorentz group $\mathcal{L}$ is conjugate (by means of some Lorentz transformation) to one of the finite subgroups $\mathcal{Y}$ of the group $\mathcal{U}$ of unitary Lorentz transformations.

Obviously, the groups of unitary Lorentz transformations are nothing other than the symmetry and antisymmetry groups of finite figures considered by A. V. Shubnikov ⁽²⁾ (among which are contained the so-called magnetic symmetry groups ⁽³⁾).

We denote by $\mathcal{G}(H_{(1)}, \dots, H_{(k)})$ the symmetry group of the tensors $H_{(1)}, \dots, H_{(k)}$, by e_1, e_2, e_3, e_4 mutually orthogonal basis vectors of Minkowski space (with e_4 regarded as a timelike vector), by g the metric tensor, by E the completely antisymmetric tensor of rank 4, and by e_k^2 the tensor product of the vector e_k with itself.

Theorem 4. Only the following continuous (or mixed) subgroups of the Lorentz group and their supergroups can be symmetry groups of tensors

$$\begin{aligned} \mathcal{G}_1(E, g), \quad \mathcal{G}_2(E, g, e_1), \quad \mathcal{G}_3(E, g, e_4), \quad \mathcal{G}_4(E, g, e_1 + e_4), \quad \mathcal{G}_5(E, g, e_1, e_2), \\ \mathcal{G}_6(E, g, e_1, e_4), \quad \mathcal{G}_7(E, g, e_1 + e_4, e_2), \\ \mathcal{G}_8(E, g, e_1 e_2 - e_2 e_1), \\ \mathcal{G}_9(E, g, e_1 + e_4, e_2(e_1 + e_4) - (e_1 + e_4)e_2). \end{aligned}$$

Here the following assertions are valid (corollaries of Theorem 4).

Let the group of proper Lorentz transformations $\mathcal{H}$ be given by the infinitesimal operators

$$\begin{aligned} X_1 &= x^2 \partial_3 - x^3 \partial_2, & X_4 &= x^1 \partial_4 + x^4 \partial_1, \\ X_2 &= x^3 \partial_1 - x^1 \partial_3, & X_5 &= x^2 \partial_4 + x^4 \partial_2, \\ X_3 &= x^1 \partial_2 - x^2 \partial_1, & X_6 &= x^3 \partial_4 + x^4 \partial_3, \end{aligned}$$

where x^i are coordinates in Minkowski space, $\partial_i = \partial/\partial x^i$.

Denote by $\mathcal{L}(X_{i_1}, \dots, X_{i_r})$ the subgroup of the Lorentz group determined by the operators $X_{i_1}, \dots, X_{i_r}$.

Theorem 5₁. If a tensor T is invariant with respect to one of the groups

$$\begin{aligned} \mathcal{H}(X_1, X_2, X_3, X_4, X_5, X_6), & \quad \mathcal{H}_2(X_6, X_4 - X_2, X_1 + X_5), \\ \mathcal{H}_1(X_3, X_6, X_4 - X_2, X_1 + X_5), & \quad \mathcal{H}_3(X_3 + \nu X_6, X_4 - X_2, X_1 + X_5), \end{aligned}$$

where $\nu > 0$, then it is invariant with respect to the group $\mathcal{G}_1(E, g)$.

Theorem 5₂. *Invariance of a tensor T with respect to the group $\mathcal{H}_4(X_5, X_6 - X_1)$ entails invariance with respect to the group*

$$\mathcal{H}_5(X_1, X_5, X_6) = \mathcal{G}_2(E, g, e_1).$$

Theorem 5₃. *From invariance of a tensor T with respect to the group $\mathcal{H}_6(X_3 + \nu X_6)$, where $\nu > 0$, it follows that it is invariant with respect to the group*

$$\mathcal{G}_8(E, g, e_1 e_2 - e_2 e_1) \supset \mathcal{H}_7(X_3, X_6).$$

3. It is obvious that the symmetry group of the tensor product of two tensors either coincides with the symmetry group of the factors, or differs from it only by transformations that change the sign simultaneously in both factors. Using this fact and Theorems 2, 3, and 4, one easily finds all subgroups of the Lorentz group that are symmetry groups of tensors, and constructs the sets of tensors that determine them.

Knowing the tensors determining a group $\mathcal{G}$, one can, from these tensors, by means of tensor multiplications and contractions, construct the elements of a tensor basis of the linear space of tensors of arbitrary rank invariant with respect to the group $\mathcal{G}$, and thus determine the structure of a tensor function of given tensor arguments, if their symmetry group is known.*

* The assumption of a certain internal symmetry of the tensor function T (i.e., symmetry with respect to some permutation of tensor indices), generally speaking, enlarges the group $\mathcal{G}_T$. The new tensor basis $T_{(1)}, \dots, T_{(q)}$ is obtained from the old basis $T_{(1)}, \dots, T_{(p)}$ by selecting elements with the prescribed internal symmetry.

The symmetry group of a given set of tensors is in most cases not difficult to find by direct calculations in accordance with its definition.

The number of linearly independent tensors of a given rank that are invariant with respect to a given group is easily determined by the methods of the theory of group representations for all finite and continuous symmetry groups of tensors.

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Note: Figure translations are in progress. See original paper for figures.

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