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Abstract

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MATHEMATICS

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ON THE QUASI-EQUIVALENCE OF BASES IN AN INFINITE CENTER OF A HILBERT SCALE

(Presented by Academician L. V. Kantorovich, March 28, 1969)

1°. Let $\{x_k\}$ be an unconditional Schauder basis in a linear topological space E . It is obvious that the system

$$y_k = \lambda_k T(x_{n_k})$$

is an unconditional basis in E for any permutation of the natural sequence $\{n_k\}$, any sequence of numbers $\lambda_k \neq 0$, and any isomorphism $T: E \rightarrow E$. The bases $\{x_k\}$ and $\{y_k\}$ are called quasi-equivalent ^(1,2).

A natural question arises: are all unconditional bases in E quasi-equivalent? This problem was solved positively for an entire class of nuclear countably normed spaces ⁽¹⁻⁴⁾. The assumption of nuclearity of the space was removed in ⁽⁵⁾ for the case of so-called finite centers of Hilbert scales**. In the present paper the quasi-equivalence of all unconditional bases in infinite centers of completely continuous Hilbert scales is established. The proof of quasi-equivalence in infinite centers differs qualitatively from the arguments of ⁽⁵⁾. This is due to the fact that Lemma 3 of ⁽⁵⁾ has no analogue for infinite centers.

2°. Every positive self-adjoint bounded operator A in a Hilbert space H generates a Hilbert scale $\{H_\alpha\}$ of Hilbert spaces H_α with scalar products

$$(x, y)_{H_\alpha} = (A^{-\alpha} x, A^{-\alpha} y),$$

where (x, y) is the scalar product in $H = H_0$ ^(1,6). We shall call the scale $\{H_\alpha\}$ completely continuous if the operator A is completely continuous. In the latter case one may assume that $H_\alpha = l_2(a_k^\alpha)$, where $a_k = 1/\lambda_k$, λ_k are the eigenvalues of the operator A , arranged in decreasing order, i.e. $a_k \uparrow \infty$.

The countably Hilbert space

$$E_\alpha(a_k) \equiv \lim_{\lambda < \alpha} \text{pr } l_2(a_k^\lambda),$$

following ⁽¹⁾, will be called the center of the scale $\{H_\lambda\}$ —finite if $\alpha < \infty$, and infinite if $\alpha = \infty$. For $x = (\xi_1, \xi_2, \dots, \xi_k, \dots) \in H_\lambda$ we shall denote

$$\|x\|_\lambda = \|x\|_{H_\lambda} = \left(\sum |\xi_k|^2 a_k^{2\lambda} \right)^{1/2}.$$

If $H_0 \supset H_1$ are two Hilbert spaces with a (completely) continuous embedding, then there exists a (completely continuous) Hilbert scale $\{G_\alpha\}$ such that $G_0 = H_0$, $G_1 = H_1$. We shall say that $\{G_\alpha\}$ is the scale generated by H_0 and H_1 , and use the notation

$$G_\alpha = (H_0)^{1-\alpha}(H_1)^\alpha.$$

We shall repeatedly use the following simple consequence of the known interpolation theorem ⁽⁷⁾.

* This question was first posed by M. G. Haplanov for the space of analytic functions.

** Here and below the terminology of paper ⁽⁵⁾ is used.

Lemma (cf. ⁽⁹⁾). Let $H_1 \supset H_2 \supset H_3 \supset H_4$ be a quadruple of Hilbert spaces with (complete) continuous embeddings. Then for any $\alpha : 0 \leq \alpha \leq 1$ the following (complete) continuous embedding holds:

$$(H_1)^{1-\alpha}(H_3)^\alpha \supset (H_2)^{1-\alpha}(H_4)^\alpha.$$

3°. We shall prove the main result.

Theorem. In the space $E = E_\infty(a_k)$, $a_k \uparrow \infty$, all unconditional bases are quasiequivalent.

Let $\{x_k\}$ be an arbitrary unconditional basis in E . It is required to show that $\{x_k\}$ is quasiequivalent to the principal basis $\{e_k\}$, consisting of the unit vectors. In other words, one has to find a permutation of the natural series $\{n_k\}$ and a numerical sequence $\sigma_k > 0$ such that the bases $\{y_k = \sigma_k x_{n_k}\}$ and $\{e_k\}$ are equivalent, i.e. the linear operator T defined by the formula

$$T\left(\sum \xi_k e_k\right) = \sum \xi_k y_k, \quad (1)$$

is an isomorphism of E onto itself.

Denote by $\{x'_k\}$ the system biorthogonal to the basis $\{x_k\}$. By Lemma 1 ⁽⁵⁾, the system of Hilbert norms

$$|x|_\lambda^{(0)} = \left(\sum |x'_k(x)|^2 \|x_k\|_\lambda^2 \right)^{1/2}, \quad \lambda < \infty,$$

is equivalent to the original system of norms $\{\|x\|_\lambda, \lambda < \infty\}$. Denote by $H_\lambda^{(0)}$ the Hilbert space obtained from E by completion with respect to the norm $\|x\|_\lambda^{(0)}$. From the equivalence of the systems of norms it follows that there exist numbers $\mu_1 < \nu_1 < \nu_2 < \mu_2$, $\lambda_1 < \lambda_2$ such that the embeddings

$$H_{\mu_1} \supset H_{\lambda_1}^{(0)} \supset H_{\nu_1} \supset H_{\nu_2} \supset H_{\lambda_2}^{(0)} \supset H_{\mu_2}$$

hold with continuous embeddings.

Take a permutation of the natural series $\{n_k\}$ and a numerical sequence $\sigma_k > 0$ such that the system $y_k = \sigma_k x_{n_k}$ satisfies the conditions

$$\|y_k\|_{\lambda_1}^{(0)} = 1, \quad \|y_k\|_{\lambda_2}^{(0)} = b_k \uparrow \infty.$$

Below it will be shown that this permutation and renormalization are the desired ones, i.e. the operator (1) is an isomorphism of E onto itself.

Applying Lemma 2 ⁽⁵⁾ first to the quadruple of spaces

$$H_{\mu_1} \supset H_{\lambda_1}^{(0)} \supset H_{\lambda_2}^{(0)} \supset H_{\mu_2}$$

and the bases $\{a_k^{-\mu_1} e_k\}$, $\{y_k\}$, and then to the quadruple of spaces

$$H_{\lambda_1}^{(0)} \supset H_{\nu_1} \supset H_{\nu_2} \supset H_{\lambda_2}^{(0)}$$

and the bases $\{y_k\}$, $\{a_k^{-\nu_1} e_k\}$, we obtain the inequalities:

$$a_k^{\nu_2 - \nu_1} \prec b_k = \|y_k\|_{\lambda_2}^{(0)} \prec a_k^{\mu_2 - \mu_1}. \quad (2)$$

Here and in what follows the notation $a_k \prec b_k$ means $a_k = O(b_k)$.

It follows from the lemma that for any $\sigma : 0 \leq \sigma \leq 1$ the continuous embeddings

$$H_{\mu_1(1-\sigma)+\nu_2\sigma} \supset G_\sigma \supset H_{\nu_1(1-\sigma)+\mu_2\sigma}, \quad G_\sigma = (H_{\lambda_1}^{(0)})^{1-\sigma} (H_{\lambda_2}^{(0)})^\sigma$$

hold.

Hence, if one takes into account that $\|y_k\|_{G_\sigma} = b_k^\sigma$, the inequalities $(\nu_1 < \nu < \nu_2)$

$$b_k^{\sigma_2} \prec \|y_k\|_\nu \prec b_k^{\sigma_1}, \quad (3)$$

follow, where

$$\sigma_1 = \sigma_1(\nu) = (\nu - \nu_1)/(\mu_2 - \nu_1), \quad \sigma_2 = \sigma_2(\nu) = (\nu - \mu_1)/(\nu_2 - \mu_1).$$

Comparing inequalities (2) and (3), we obtain:

$$a_k^{\gamma_2} \prec \|y_k\|_\nu \prec a_k^{\gamma_1}, \quad (4)$$

where

$$\gamma_1 = \sigma_1(\mu_2 - \mu_1), \quad \gamma_2 = \sigma_2(\nu_2 - \nu_1).$$

Now take an arbitrary $\lambda > \lambda_2^{(0)}$ and $\tau = \tau(\lambda) \geq \nu_2$, $\chi = \chi(\lambda)$ such that the continuous embeddings hold:

$$H_\tau \supset H_\lambda^{(0)} \supset H_\chi$$

and $\tau(\lambda) \uparrow \infty$ as $\lambda \uparrow \infty$.

Let $\{y_{n_k}\}$ be a rearrangement of the basis $\{y_k\}$ in decreasing order of the norms in $H_\lambda^{(0)}$, i.e.,

$$\|y_{n_k}\|_\lambda^{(0)} = s_k \uparrow \infty.$$

Application of Lemma 2⁽⁵⁾, analogously to the preceding one, gives the inequalities:

$$a_k^{\tau-\nu_1} \prec \|y_{n_k}\|_\lambda^{(0)} = s_k \prec a_k^{\chi-\mu_1}. \quad (5)$$

As before, after applying the lemma of the present paper we obtain the estimates ($\nu_1 < \nu < \tau$):

$$s_k^{\tilde{\sigma}_2} \prec \|y_{n_k}\|_\nu \prec s_k^{\tilde{\sigma}_1}, \quad (6)$$

where $\tilde{\sigma}_1 = \tilde{\sigma}_1(\nu, \lambda) = (\nu - \nu_1)/(\chi - \nu_1)$,
 $\tilde{\sigma}_2 = \tilde{\sigma}_2(\nu, \lambda) = (\nu - \mu_1)/(\tau - \mu_1)$.

The following inequalities are the result of comparing inequalities (5) and (6):

$$a_k^{\tilde{\gamma}_2} \prec \|y_{n_k}\|_\nu \prec a_k^{\tilde{\gamma}_1}, \quad (7)$$

where $\tilde{\gamma}_1 = \tilde{\sigma}_1(\chi - \mu_1)$, $\tilde{\gamma}_2 = \tilde{\sigma}_2(\tau - \nu_1)$.

Comparing inequalities (4) and (7), we obtain:

$$a_{n_k}^{\gamma_2/\tilde{\gamma}_1} \prec a_k \prec a_{n_k}^{\gamma_1/\tilde{\gamma}_2}.$$

Together with inequality (5) this gives

$$a_{n_k}^{\rho_2} \prec \|y_{n_k}\|_\lambda^{(0)} \prec a_{n_k}^{\rho_1}, \quad (8)$$

where

$$\rho_1 = \rho_1(\lambda) = \frac{\gamma_1}{\tilde{\gamma}_2}(\chi - \mu_1) = \frac{(\nu - \nu_1)(\mu_2 - \mu_1)(\tau - \mu_1)}{(\mu_2 - \nu_1)(\nu - \mu_1)(\tau - \nu_1)}(\chi - \mu_1),$$

$$\rho_2 = \rho_2(\lambda) = \frac{\gamma_2}{\tilde{\gamma}_1}(\tau - \nu_1) = \frac{(\nu - \mu_1)(\nu_2 - \nu_1)(\chi - \nu_1)}{(\nu_2 - \mu_1)(\nu - \nu_1)(\chi - \mu_1)}(\tau - \nu_1).$$

Obviously, as $\lambda \uparrow \infty$,

$$\rho_2(\lambda) \uparrow \infty. \quad (9)$$

Since $\{n_k\}$ is a rearrangement of the natural sequence, formula (8) may be rewritten in the form

$$C_2 a_k^{\rho_2} \leq \|y_k\|_\lambda^{(0)} \leq C_1 a_k^{\rho_1}, \quad C_1 = C_1(\lambda) < \infty, \quad C_2 = C_2(\lambda) > 0. \quad (10)$$

It is now easy to establish that the operator T , defined by formula (1), is an isomorphism.

I take this opportunity to thank B. S. Mityagin, who drew my attention to the fact that a number of essential points in the proof of Theorem 1 in paper ⁽⁵⁾ had been used earlier (for another purpose) in ⁽²⁾ (see the proof of Theorem 11).

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