

# ON THE ROTATION OF MULTIVALUED VECTOR FIELDS

Yu. G. Borisovich, B. D. Gel'man, È. Muhamadiev, V. V.  
Obukhovskii

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## Abstract

## Full Text

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## MATHEMATICS

Yu. G. BORISOVICH, B. D. GELMAN, E. MUKHAMADIEV, V. V. OBUKHOVSKII

# ON THE ROTATION OF MULTIVALUED VECTOR FIELDS

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1. A mapping  $F$  acting in a Banach space  $X$  is called multivalued if to each point  $x \in X$  there corresponds a subset  $F(x) \subset X$ . A point  $x_0 \in X$  is called a fixed point of the multivalued mapping  $F$  if  $x_0 \in F(x_0)$ . The problem of the existence of fixed points for multivalued mappings is of interest for various branches of mathematics (dynamical systems without uniqueness, game theory, etc.). A number of works are devoted to it <sup>(1-5)</sup>.

In the present article, for multivalued mappings with convex images, we introduce a new topological invariant—the notion of rotation of the corresponding vector field, closely connected with fixed-point theory. A number of properties analogous to the properties of the rotation of single-valued vector fields are established (on these questions see, for example, <sup>(4)</sup>); applications to fixed-point theorems are given.

2. We first consider the finite-dimensional space  $X = R_n$ . Let  $\bar{G}$  be the closure of an open set  $G$  with triangulable boundary  $\Gamma$ . Suppose that a mapping  $F$  with convex images is defined on  $\Gamma$  and has the following properties: 1) it is closed (i.e., the graph

$$\bigcup_{x \in \Gamma} (x, F(x))$$

is closed in  $X \times X$ ); 2) it is compact (i.e.,

$$\overline{\bigcup_{x \in \Gamma} F(x)}$$

is compact). Such mappings will be called completely continuous mappings, bearing in mind that this definition is to be extended below also to mappings in Banach spaces.

A multivalued mapping defines a multivalued vector field on the boundary  $\Gamma$  by the formula  $\Phi(x) = x - F(x)$  (in the sense of the algebraic difference of two sets), which we shall also call completely continuous. If  $x_0 \in F(x_0)$ , then the

image  $\Phi(x_0)$  contains  $\Theta$ . Such a point  $x_0$  will be called a special point of the multivalued vector field.

**Lemma 1.** *Let  $F$  be a completely continuous multivalued mapping  $\Gamma \rightarrow R_n$ , and let  $x_j \rightarrow x$ ,  $x_j \in \Gamma$ . Then for every  $\varepsilon > 0$  there is an  $n_0$  such that  $F(x_j) \subset F^\varepsilon(x)$ ,  $j \geq n_0$ , where  $F^\varepsilon(x)$  is the  $\varepsilon$ -neighborhood of  $F(x)$ .*

We make a simplicial subdivision of the boundary  $\Gamma$  and choose an arbitrary simplex  $S_p$  of this subdivision. Consider the  $F$ -images  $F(y_p^1), \dots, F(y_p^k)$  of its vertices. We fix arbitrarily points  $z^r \in F(y_p^r)$  ( $r = 1, \dots, k$ ) and span over them the convex hull  $\Delta$ . Define a continuous single-valued mapping  $f : S_p \rightarrow \Delta$  by the equality

$$f\left(\sum_{r=1}^k \lambda_r y_p^r\right) = \sum_{r=1}^k \lambda_r z^r \quad \left(\lambda_r \geq 0, \sum_{r=1}^k \lambda_r = 1\right).$$

Thus we have constructed a continuous single-valued approximation  $f : \Gamma \rightarrow R_n$  of the multivalued mapping  $F$ .

**Lemma 2.** *Under the condition that  $\Phi x$  has no special points on  $\Gamma$ , and for sufficiently fine simplicial subdivisions of the boundary  $\Gamma$ , the vector field  $\varphi(x) = x - f(x)$  also has no special points on  $\Gamma$ .*

**Proof.** Let  $\varepsilon > 0$  be the smallest distance between the points  $x$  and their  $F$ -images, existing by Lemma 1. Let  $V(\Theta) =$

neighborhood of zero of radius  $\eta < \varepsilon/2$ . Then  $V(x) \cap V(F(x)) = \emptyset$  for every  $x \in \Gamma$ . By  $U(x)$  we denote the largest neighborhood of the point  $x$  such that  $F(y) \subset V(F(x))$ , if  $y \in U(x)$  (its existence follows from Lemma 1). Let  $\delta$  be a Lebesgue number of the cover  $\{U(x) \cap V(x)\}$  of the boundary  $\Gamma$ ; choose a simplicial subdivision of  $\Gamma$  with the diameter of the simplexes less than  $\delta/2$ . Then each simplex  $S_p$  belongs to some neighborhood  $U(x) \cap V(x)$ , and the  $F$ -images of its vertices lie in  $V(F(x))$ . Consequently, the  $f$ -image of the simplex  $S_p$  lies in  $V(F(x))$ , whence the assertion of the lemma follows.

Below it is assumed that the space  $R_n$  is oriented. We formulate the basic

**Definition 1.** The **rotation**  $\gamma(\Phi, \Gamma)$  of a multivalued field  $\Phi$  on the boundary  $\Gamma$  will mean  $\gamma(\varphi, \Gamma)$  of a single-valued vector field  $\varphi$ , where  $\varphi$  is constructed on a sufficiently fine simplicial subdivision of  $\Gamma$ .

It can be shown that the definition does not depend on the choice of the simplicial subdivision, the points  $z^r$ , and the radius of the separating neighborhood  $V$ .

Let us note that another approach is also possible, based on the notion of the degree of a multivalued mapping of the boundary  $\Gamma$  into the sphere  $S^{n-1}$ . First define the degree of such a mapping  $\Psi$ , which is assumed to be closed and for which the spherical convex hull of each image  $\Psi(x)$  does not contain diametrically opposite points. In this case one can construct a single-valued

approximation of the mapping  $\Psi$ . To this end, for each point  $x \in \Gamma$  we fix a neighborhood  $U(x)$  so that the  $\Psi$ -images of the points  $y \in U(x)$  belong to the  $\varepsilon_x$ -inflation of the image  $\Psi(x)$ , where  $\varepsilon_x$  is so small that the  $\varepsilon_x$ -inflation contains no diametrically opposite points. The cover  $\{U(x)\}$  determines a Lebesgue number  $\delta$ , and, if the triangulation is sufficiently fine, then a single-valued approximation  $\psi$  is constructed, as in the first case, affinely on each simplex.

**Definition 2.** The degree of the mapping  $\psi : \Gamma \rightarrow S^{n-1}$  under a sufficiently fine triangulation of  $\Gamma$  will be called the **degree of the multivalued mapping**  $\Psi : \Gamma \rightarrow S^{n-1}$ .

It is not difficult to show that this definition is correct.

**Definition 3.** The **rotation of a multivalued vector field**  $\Phi$  is called the degree of the multivalued mapping

$$\Psi(x) = \left\{ y : y = \frac{z}{\|z\|}, z \in \Phi(x) \right\}$$

of the boundary  $\Gamma$  into  $S^{n-1}$ .

Definitions 1 and 3 are equivalent.

3. A number of properties known for the rotation of a single-valued field <sup>(4)</sup> carry over to the notion of the rotation of a multivalued vector field.

Multivalued vector fields  $\Phi_0$  and  $\Phi_1$  will be called homotopic on  $\Gamma$  if on  $\Gamma \times I$ , where  $I = [0, 1]$ , there exists a completely continuous multivalued vector field  $\Phi(x, t)$  such that  $\Theta \in \Phi(x, t)$ ,  $x \in \Gamma$ ,  $t \in I$ ,  $\Phi(x, 0) = \Phi_0(x)$ ,  $\Phi(x, 1) = \Phi_1(x)$ .

**Theorem 1.** If  $\Phi_0$  is homotopic to  $\Phi_1$  on  $\Gamma$ , then  $\gamma(\Phi_0, \Gamma) = \gamma(\Phi_1, \Gamma)$ .

The assertion of the theorem follows from the fact that in a neighborhood of each point  $t_0 \in I$  one can construct a common single-valued approximation for  $\Phi(x, t)$ .

The following proposition may be regarded as a general criterion for the existence of a fixed point.

**Theorem 2.** Let a multivalued completely continuous mapping  $F$  be given on the closed domain  $\overline{G}$ , with no fixed points on the boundary  $\Gamma$ , and let  $\gamma(\Phi, \Gamma) \neq 0$ , where  $\Phi$  is the corresponding vector field. Then there exists a fixed point  $x_0 \in G$  of the mapping  $F$ .

From this theorem one easily obtains a generalization of S. Kakutani's theorem <sup>(2)</sup>.

**Theorem 3.** If a multivalued completely continuous mapping  $F$  is given on the closed ball  $T \subset R_n$  and, for every  $x$  belonging to the boundary  $S^{n-1}$  of the ball  $T$ , the condition  $F(x) \cap T = \emptyset$  is satisfied, then there exists a fixed point  $x_0 \in T$  of the mapping  $F$ .

Indeed, in this case the rotation  $\gamma(\Phi, \Gamma)$  of the field  $\Phi$  is equal to one.

Multivalued mappings, besides isolated fixed points, may have connected sets of fixed points. Both will be called fixed elements. A fixed element  $x_*$  will be called isolated if some neighborhood of it does not intersect other fixed elements. The rotation of a multivalued vector field on the boundaries of such neighborhoods is constant; we shall call it the index  $\gamma(x_*)$  of the fixed element.

**Theorem 4.** If, on the closed domain  $\overline{G}$ , a multivalued completely continuous vector field  $\Phi$  has a finite number of isolated fixed elements  $x_1, \dots, x_k$  and has no special points on the boundary  $\Gamma$ , then

$$\gamma[\Phi, \Gamma] = \sum_{i=1}^k \gamma(x_k).$$

4. In conclusion we consider completely continuous multivalued vector fields  $\Phi$  in a Banach space (we retain the definition of Sec. 2). In this case Lemmas 1 and 2 remain valid. We construct a multivalued finite-dimensional approximation  $\Phi_m$  of the field  $\Phi$ .

Let  $V$  be the closed neighborhood of zero of radius  $\rho < \varepsilon/2$ . Choose in the range  $F(\Gamma)$  a  $\rho$ -net  $\{y_1, \dots, y_m\}$ , and denote by  $K_m$  the convex hull, and by  $E_m$  the linear hull, of the net  $\{y_1, \dots, y_m\}$ . We shall assume that  $G \cap E_m$  has a polyhedral boundary  $\Gamma_m$  for each  $E_m$ . Consider the multivalued vector field  $\Phi_m(x) = (\Phi(x) + V) \cap K_m$  on  $\Gamma_m$ . This field, as can be shown, has no special points and satisfies the conditions of Sec. 2.

**Theorem 5.** The rotation  $\gamma[\Phi_m, \Gamma_m]$  does not depend on the choice of the approximating space  $E_m$  or on the choice of  $\rho < \varepsilon/2$ .

It is therefore natural to give the following definition:

$$\gamma\{\Phi, \Gamma\} = \gamma\{\Phi_m, \Gamma_m\}.$$

The theorems proved in Sec. 3 are also valid in the case of a Banach space, in the very same formulations. We give the formulation of a theorem generalizing I. Glicksberg's theorem<sup>1</sup>.

**Theorem 6.** If a multivalued completely continuous field  $\Phi(x)$  is defined on the closed ball  $T$  of a Banach space, has no special points on the boundary  $\Gamma$ , and satisfies the condition  $F(x) \cap T = \emptyset$ ,  $x \in \Gamma$ , then  $\gamma[\Phi, \Gamma] = 1$ ; consequently, there exists a fixed point.

Applying the general schemes known in the theory of rotations of single-valued vector fields, one can also prove more general theorems on the existence of fixed points.

**Remark.** The condition of complete continuity of the mapping  $F$  in the preceding constructions can be somewhat weakened: the assumption of compactness of

the image  $F(\Gamma)$  may be replaced by the condition of the existence of a compact set  $K$  intersecting each image  $F(x)$ , where  $x \in \overline{G}$ .

Voronezh State University

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*Note: Figure translations are in progress. See original paper for figures.*

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