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Abstract

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ON THE MINIMIZATION OF A QUADRATIC FUNCTIONAL GENERATED BY A LINEAR OPERATOR IN NORMED SPACES

Some problems in the theory of optimal control lead to finding the minimum of a functional of the form

$$\Phi(f) = \|Af - g\|^2, \quad f \in D(A), \quad (1)$$

where f is an element of a Banach space B ; g is a given element of a Hilbert space H ; A is a linear operator acting from B into H ; $D(A)$ is the domain of definition of the operator A .

Let us denote by $R(A)$ the range of values of the operator A . For example, the functional

$$\Phi(f) = \|A_1 f - g_1\|^2 + \|A_2 f - g_2\|^2, \quad (2)$$

where f is an element of a Banach space B ; g_1 is a given element of a Hilbert space H_1 ; g_2 is a given element of a Hilbert space H_2 ; A_1 is a linear operator acting from B into H_1 , with domain of definition $D(A_1)$; A_2 is a linear operator acting from B into H_2 , with domain of definition $D(A_2)$, is reduced to (1).

Indeed, consider the Hilbert space

$$H = H_1 \times H_2 = \{\varphi, \varphi = [\varphi_1, \varphi_2], \varphi_1 \in H_1, \varphi_2 \in H_2,$$

$$\langle \varphi', \varphi'' \rangle = (\varphi'_1, \varphi''_1) + (\varphi'_2, \varphi''_2)\}.$$

Introduce a linear operator A , acting from B into H , and defined as follows:

$$Af = [A_1 f, A_2 f], \quad \text{where } f \in D = D(A_1) \cap D(A_2).$$

Then the functional (2) is reduced to the form (1).

To solve the problem of minimizing the functional (1), consider the special case when A is a monomorphism, i.e., maps B into H in a one-to-one manner. Then the following obvious equality holds:

$$\min_{f \in D(A)} \|Af - g\|^2 = \min_{h \in R(A)} \|h - g\|^2. \quad (3)$$

The problem of minimizing the simplest functional $\Phi(f) = \|f - g\|^2$ in a Hilbert space has been investigated in the following formulation: find an element f_0 from some manifold G for which

$$\min_{f \in G} \Phi(f) = \|f_0 - g\|^2, \quad (4)$$

where, generally speaking, g does not belong to G (obviously, if $g \in G$, then $f_0 = g$ and $\min \Phi(f) = 0$).

Theorem 1. If G is a linear manifold, then the solution of the problem is unique; if, in addition, G is a closed manifold (i.e., G is a subspace of the space H), then there also exists a solution, and if f_0 is a solution of the problem, then the element

$f_0 - g \perp G$, i.e., the equality holds

$$(f_0 - g, f) = 0 \quad \text{for all } f \in G. \quad (5)$$

Theorem 2. If in the expression of functional (2) the operator A_2 has a bounded inverse, and A_1 is a closed operator, then the minimization problem for functional (2) has a solution, and moreover a unique one.

Proof. For the proof it suffices to establish that in the corresponding functional (1) the operator A is a monomorphism and $R(A)$ is a subspace of the space H .

First let us prove that the operator A is a monomorphism. For this it is necessary to prove that from $Af = 0$ it follows that $f = 0$. Let $Af = 0$. By definition, then $A_1f = 0$ and $A_2f = 0$. Since A_2 is a monomorphism, $f = 0$, as was required to prove.

Further, from the linearity of the operator A follows the linearity of the manifold $R(A)$. It remains to show that $R(A)$ is a closed manifold. For this purpose take an arbitrary sequence $h_\nu \in R(A)$ converging to some element h . It is necessary to prove that $h \in R(A)$. This is proved as follows. In view of the monomorphism, to each h_ν there corresponds a unique f_ν such that $h_\nu = Af_\nu$, i.e. if $h_\nu = (h'_\nu, h''_\nu)$, then $h'_\nu = A_1f_\nu$, $h''_\nu = A_2f_\nu$. Then h'_ν converge to h' and $h''_\nu \rightarrow h''$, respectively. Further we have the sequence $f_\nu = A_2^{-1}h''_\nu$, which converges to the element $f = A_2^{-1}h''$. Then $h' = A_1f$ by virtue of the closedness

of A_1 , since $f_\nu \rightarrow f$ and $A_1 f_\nu \rightarrow h'$, so $h' = A_1 f$. From the latter and from $h'' = A_2 f$ it follows that A is closed and $h = Af$, as was required to prove.

Then, by the classical theorem, there exists f_0 , $Af_0 = h_0 \in R(A)$, and it is unique, minimizing functional (2). This completes the proof of the theorem.

Quadratic functionals considered, for example, in (1) by Bellman et al. in a finite-dimensional Hilbert space, and by Z. Khalilov and E. Aslanov ^(2,3) in a Hilbert space, are particular cases of functional (2). The theorem proved above is general in the theory of partial equations considered in works ⁽¹⁻³⁾. Its applications are especially effective in the investigation of optimal-control problems for partial differential equations.

For finding the existing unique minimizing element f_0 , under additional conditions one can construct a certain functional equation. Indeed, if we apply (5), then

$$(Af_0 - g, Af) = 0$$

for all $f \in D(A)$.

Let $D(A)$ be dense in B . Then from $(A^*(Af_0 - g), f) = 0$ we have

$$A^*Af_0 = A^*g. \quad (6)$$

It can be shown that (6) is equivalent to the equation:

$$A_1^*A_1f_0 + A_2^*A_2f_0 = A_1^*g_1 + A_2^*g_2. \quad (7)$$

Let us now consider another characteristic functional:

$$\Phi(f_1, f_2) = \|A_1'f_1 + A_1''f_2 - g_1\|^2 + \|A_2'f_1 + A_2''f_2 - g_2\|_2^2, \quad (8)$$

where $f_1 \in B_1$, $f_2 \in B_2$, $g_1 \in H_1$, $g_2 \in H_2$, $A_1' : B_1 \rightarrow H_1$, $A_1'' : B_2 \rightarrow H_1$, $A_2' : B_1 \rightarrow H_2$,

$$A_2'' : B_2 \rightarrow H_2.$$

It is not difficult to see that (8) can also be reduced to (2). Indeed, introduce the Banach space $B = B_1 \times B_2$ with elements $f = (f_1, f_2)$ with norm $\|f\|^2 = \|f_1\|^2 + \|f_2\|_2^2$ and operators $A_1f = A_1'f_1 + A_1''f_2$, $A_2f = A_2'f_1 + A_2''f_2$, $Af = [A_1f, A_2f]$. Then (8) takes the form (2), which in turn reduces to (1).

Thus, the main theorem can be reformulated also for (8). If A_1' is a bounded linear operator, A_1'' is a closed li-

linear operator and A_2 is a linear operator having a bounded inverse, then, according to the theorem, the minimization problem for the functional (8) has a solution, and moreover a unique one.

To obtain equation (7) and other functional equations, the following formulas are used:

$$\text{if } Af = [A_1f, A_2f], \quad \text{then } A^*\varphi = A_1^*\varphi_1 + A_2^*\varphi_2; \quad (9)$$

$$\text{if } Af = A_1f_1 + A_2f_2, \quad \text{then } A^*\varphi = [A_1^*\varphi, A_2^*\varphi]. \quad (10)$$

Then, if we are dealing with the functional (8), the functional equation of type (7) for the functional (8), on the basis of (9) and (10), has the form (see (6))

$$A_1^*(A_1'f_1^0 + A_1''f_2^0) + A_2^*(A_2'f_1^0 + A_2''f_2^0) = A_1^*g_1 + A_2^*g_2,$$

$$A_1'^*(A_1'f_1^0 + A_1''f_2^0) + A_2'^*(A_2'f_1^0 + A_2''f_2^0) = A_1'^*g_1 + A_2'^*g_2.$$

Let us note that, in an analogous way, one can consider a functional of a more general form:

$$\Phi(f_1, f_2, \dots, f_m) = \sum_{i=1}^n \left\| \sum_{j=1}^m A_{ij}f_j - g_i \right\|_i^2,$$

where $f_j \in D(A_{ij}) \subset B_j$, $g_i \in H_i$ for any i , which, by the corresponding grouping, is reduced to a functional of the form (2). It is of interest to consider the corresponding problem for operators mapping into a Banach space.

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Note: Figure translations are in progress. See original paper for figures.

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