

# ON PERFECT IRREDUCIBLE MAPPINGS OF COMPLETELY REGULAR SPACES

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**Abstract**

**Full Text**

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*MATHEMATICS*

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## ON PERFECT IRREDUCIBLE MAPPINGS OF COMPLETELY REGULAR SPACES

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All mappings are assumed to be continuous.

**Definition 1.** Let there be an inverse system

$\Sigma = \{X_\alpha, \pi_{\alpha_2}^{\alpha_1}\}$  ( $\alpha \in A$ ) of topological spaces  $X_\alpha$  with continuous mappings  $\pi_{\alpha_2}^{\alpha_1}$ , constructed on the directed set  $A = \{\alpha\}$ . Suppose that in this set there is a minimal element  $\alpha_0$ , for which  $\alpha \geq \alpha_0$  for every  $\alpha$ . Then for every space  $X_\alpha$  ( $\alpha \in A$ ) there exists a mapping onto  $X_{\alpha_0}$ . The limit of the system  $X$  can also be naturally mapped onto  $X_{\alpha_0}$ . We shall say that the mapping  $\pi : X \rightarrow X_{\alpha_0}$  is **decomposed into a system of mappings**  $\pi_{\alpha_0}^\alpha : X_\alpha \rightarrow X_{\alpha_0}$ .

**Definition 2.** A mapping  $f : X \rightarrow Y$  will be called **elementary** if in  $X$  there is an open everywhere dense set  $U$  on which  $f$  is a homeomorphism.

### § 1. The extension $d_{XY}$ and its properties

Let  $X$  and  $Y$  be locally bicomact spaces;  $f : X \rightarrow Y$  a perfect irreducible mapping;  $bX$  a bicomact extension of the space  $X$ ;  $\alpha$  a proximity space on  $X$  generated by the extension  $bX$ . Define on  $Y$  a proximity space  $\beta$ :  $A\bar{\beta}B$  if and only if

$$(f^{-1}A)\alpha(f^{-1} - B).$$

All the axioms of a proximity space are easily verified.

The proximity space  $\beta$  corresponds to a bicomact extension of the space  $Y$ , which we shall denote by  $d_{bX}Y$ , or, when it is clear which  $bX$  and  $f$  are meant, simply by  $dY$ .

We establish some properties of the extension.

**Theorem 1.** *If the mapping  $f : X \rightarrow bY$  can be extended to a mapping  $g : bX \rightarrow bY$ , then there exists a mapping  $h : d_{bX}Y \rightarrow bY$ , identical on  $Y$ .*

The proof follows easily from the definition of  $d_{bX}Y$ .

**Theorem 2.** Let  $g$  be the embedding  $Y \rightarrow d_{bX}Y$ ; then the mapping  $fg : X \rightarrow dY$  can be extended to a mapping  $bX \rightarrow dY$ .

**Part of the proof of Theorem 2.** Let  $x \in bX$ , and let  $\delta$  be the system of all neighborhoods of the point  $x$ :  $\delta = \{S_\alpha\}_{\alpha \in A}$ . Put  $T_\alpha = f^\#(S_\alpha \cap X)$ —this is a nonempty open set by virtue of the irreducibility and closedness of the mapping  $f$ . If there exists a finite family of sets  $T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}, \dots, T_{\alpha_n}$ , then

$$f^{-1}\left(\bigcap_{k=1}^n T_{\alpha_k}\right) \subset X \cap \left(\bigcap_{k=1}^n S_{\alpha_k}\right).$$

From the centeredness of the system  $\delta$  follows the centeredness of the system  $\zeta = \{T_\alpha\}_{\alpha \in A}$ . Since  $dY$  is bicomact,  $\zeta$  has a cluster point in  $dY$ . We prove that this point is unique.

1. Suppose that this is not so:  $y_1 \in [\zeta]$ ,  $y_2 \in [\zeta]$ ,  $y_1 \neq y_2$ . (By  $[\zeta]$  we denote the set of all cluster points of the filter  $\zeta$ .)
2. At the points  $y_1$  and  $y_2$  choose neighborhoods  $U_1$  and  $U_2$  with disjoint closures in  $d_{bX}$ .
3.  $U_1 \cap Y$  and  $U_2 \cap Y$  are far in the proximity space  $\beta$ .
4. Denote  $W_1 = f^{-1}(U_1)$ ,  $W_2 = f^{-1}(U_2)$ .
5.  $W_1 \alpha W_2$ , according to the definition of the proximity  $\beta$ .
6. Hence it follows that  $[W_1]_{bX} \cap [W_2]_{bX} = \Lambda$ .
7.  $U_1 \cap T = \Lambda$ ,  $U_2 \cap T = \Lambda$  for every  $T \in \xi$ .
8. Hence, and from the definition of the filter  $\xi$ , it follows that  $W_1 \cap S = \Lambda$ ,  $W_2 \cap S = \Lambda$  for every  $S \in \delta$ .
9. Since  $\delta$  is the filter of all neighborhoods of the point  $x$ ,  $[W_1] \ni x$ ,  $[W_2] \ni x$ .
10. Items 9 and 6 contradict each other, as was required to prove.

Assign to each point  $x \in X$  the unique limit point  $y$  of the filter  $\zeta(x)$ . The mapping  $g$  thus constructed will be the desired mapping  $bX \rightarrow dY$ .

**§ 2. Decomposition of an irreducible mapping into elementary mappings.** Let  $f : X \rightarrow Y$  be an irreducible mapping of the bicomactum  $X$  onto the bicomactum  $Y$ . Let  $\mathfrak{B}$  be a base in  $X$  of cardinality  $\tau$ ;  $U_1$  and  $U_2$  are elements of the base;  $[U_1] \cap [U_2] = \Lambda$ . Put

$$F(U_1, U_2) = f[U_1] \cap f[U_2].$$

Since the mapping  $f$  is irreducible,  $F$  is nowhere dense.  $Y \setminus F$  is a locally bicomact space; it is a perfect irreducible image of its full preimage  $P \subset X$ .

By Theorem 2 there exists a bicomact extension  $d_X(Y \setminus F)$  with the natural mapping

$$g : X \rightarrow d_X(Y \setminus F).$$

**Lemma.**  $g(U_1) \cap g(U_2) = \Lambda$ .

**Proof of the lemma.** The open sets  $U_1 \cap f^{-1}(Y \setminus F)$  and  $U_2 \cap f^{-1}(Y \setminus F)$  have disjoint closures in  $X$ . Their marked kernels are far in  $\alpha$ ; consequently, the images  $V_1$  and  $V_2$  of these kernels in  $Y \setminus F$  are far in the proximity space  $\beta$ . Hence it follows that

$$[V_1]_d \cap [V_2]_d = \Lambda,$$

where  $d = d(Y \setminus F)$ .

It is easy to verify that

$$g(U_i) \subseteq [V_i]_{d(Y \setminus F)} \quad (i = 1, 2),$$

as was required to prove.

Since  $\mathfrak{B}$  is a base of the bicomactum  $X$  of cardinality  $\tau$ , for any two points  $x_1$  and  $x_2$  of  $X$  one can choose neighborhoods  $U_1$  and  $U_2$  with disjoint closures. It follows from the lemma that the mapping

$$g : X \rightarrow d_X[Y \setminus F(U_1, U_2)]$$

sends the points  $x_1$  and  $x_2$  to distinct points  $r_1$  and  $r_2$  in the space  $d_X$ .

Let  $\eta$  be the system of all sets  $F(U_1, U_2)$ , where  $U_1, U_2 \in \mathfrak{B}$ , and let  $\vartheta$  be the system of all finite unions of sets from  $\eta$ . Obviously, the cardinality of the system  $\vartheta$  is equal to  $\tau$ . The system of all extensions  $d_X(Y \setminus P)$  ( $P \in \vartheta$ ) forms an inverse spectrum  $\Sigma(\mathfrak{B})$  with natural projections

$$d_X(Y \setminus P_1) \rightarrow d_X(Y \setminus P_2),$$

when  $P_1 \subseteq P_2$ . (For the construction of the spectrum see <sup>(2)</sup>.) Denote the limit of the spectrum  $\Sigma(\mathfrak{B})$  by  $Y$ .

**Theorem 3.**  $X = \lim \Sigma(\mathfrak{B})$ .

**Proof.** Consider the system of mappings

$$\{g_\lambda : X \rightarrow d_X(Y \setminus F_\lambda)\}.$$

It generates a mapping

$$\{g : X \rightarrow d_X(Y \setminus F_\lambda)\}.$$

For any two points of  $X$  there is a mapping that sends them to distinct points in  $d_X(Y \setminus F_{\lambda(x_1, x_2)})$ . From the definition of the limit of a spectrum it follows at once that  $\tilde{g}(x_1) \neq \tilde{g}(x_2)$ . This means that the mapping  $g$  is one-to-one, i.e. a homeomorphism, as was required to prove.

From Theorem 3 it follows:

**Theorem 4.** *An irreducible mapping of a bicomcompactum  $X$  of weight  $\tau$  onto a bicomcompactum  $Y$  decomposes into a spectrum of  $\tau$  elementary mappings.*

If  $X$  and  $Y$  are compacta, we obtain:

**Theorem 5.** *An irreducible mapping of a compactum  $X$  onto a compactum  $Y$  is the product of a countable number of elementary mappings.*

**Corollary of Theorem 5.** *If  $X \rightarrow Y$  is an irreducible mapping of a compactum  $X$  onto a compactum  $Y$ , then in  $Y$  there is everywhere a dense set of type  $G_\delta$  of points of one-to-oneness.*

**§ 3. Decomposition of an irreducible mapping of a bicomcompactum into a transfinite spectrum of elementary mappings.** Let  $f : X \rightarrow Y$  be an irreducible mapping of a bicomcompactum  $X$  onto a bicomcompactum  $Y$ . Analogously to how this was done in <sup>(2)</sup>, choose in  $Y$  an open everywhere dense set  $U$  so that  $Y_1 = d_{XU}$  differs from  $Y$ . In  $Y$  choose an everywhere dense open set  $U_1$  and construct  $d_{XU_1} = Y_2$ , and so on, by transfinite induction. If  $\alpha$  is a limit ordinal, then as  $Y_\alpha$  we take the limit of the spectrum of all  $Y_\beta$  for ordinals  $\beta < \alpha$ .

The transfinite process of constructing the spaces  $Y_\alpha$  necessarily terminates at some cardinal number  $\Omega(X, Y)$ , since the weight of the bicomcompactum  $X$  does not exceed  $2^\tau$ , where  $\tau$  is the weight of the bicomcompactum  $Y$ . Obviously, the limit of the spectrum of the spaces  $\{Y_\alpha\}_{\alpha \leq \Omega}$  is an irreducible image of the bicomcompactum  $X$ . It coincides with  $X$ , since otherwise one could find an open everywhere dense set  $\tilde{U} \subset \tilde{Y}$  with the property  $d_X \tilde{U} \neq \tilde{Y}$ . This contradicts the fact that the process terminates at the space  $\tilde{Y}$ .

We have proved the following proposition:

**Theorem 6.** *Every irreducible mapping of a bicomcompactum  $X$  onto a bicomcompactum  $Y$  decomposes into a transfinite spectrum of mappings  $\pi_{\alpha_2}^{\alpha_1}$ , in which every mapping  $\pi_{\alpha}^{\alpha+1}$  is elementary.*

**§ 4. Incomplete absolutes.** Let  $X$  be a bicomcompactum and let  $\eta$  be some system of its closed nowhere dense subsets, possessing the following properties: 1) if  $F_1 \in \eta$ ,  $F_2 \in \eta$ , then  $F_1 \cup F_2 \in \eta$ ; 2) if  $F_1 \subset F_2$ ,  $F_2 \in \eta$ , then  $F_1 \in \eta$ . We shall call the space  $X$ , together with the system  $\eta$  specified in it, a pair.

Let  $f : Z \rightarrow X$  be an irreducible mapping, and suppose that for any two open sets  $U_1$  and  $U_2$  in  $Z$  whose closures are disjoint, one has

$$f[U_1] \cap f[U_2] \in \eta.$$

We shall say that the mapping  $f$  is generated by the system  $\eta$ . If  $F \in \eta$ , then  $f^{-1}(F)$  is nowhere dense in  $Z$ . The mapping  $f$  and the system  $\eta$  generate in  $Z$  the system

$$\zeta = f^{-1}(\eta) = \{f^{-1}(F) \setminus F \in \eta\}.$$

Suppose that, along with the mapping  $f$ , there exists a mapping  $g : Z \rightarrow Y$ , and moreover  $g^{-1}(g[U_1] \cap g[U_2]) \in \zeta$  ( $[U_1] \cap [U_2] = \Lambda$ ). The mapping  $g$  generates in the space  $Y$  the system  $\vartheta = \{g(F), F \in \zeta\}$ . We shall say that the pair  $(X, \eta)$  is coabsolute with the pair  $(Y, \vartheta)$ . It is easy to see that the mapping  $g$  is generated by the system  $\vartheta$ .

**Assertion.** *The relation of coabsoluteness is an equivalence relation.*

Reflexivity and symmetry are obvious. Let us verify transitivity. Suppose  $X(\eta) \sim Y(\zeta)$ ,  $Y(\zeta) \sim Z(\vartheta)$ . This means that there exist four mappings

$$P \rightarrow X, \quad Q \rightarrow Z, \quad P \rightarrow Y, \quad Q \rightarrow Y.$$

By Theorem 4,  $P$  and  $Q$  are limits of the spectrum of the spaces  $b(Y \setminus F)$ ,  $F \in \eta$ . To each  $F$  there correspond two bicomact extensions  $b_p(Y \setminus F)$  and  $b_q(Y \setminus F)$ .

Taking the extension  $\beta(Y \setminus F)$  for each  $F$  and constructing a spectrum from them, we obtain in the limit a space  $R$ , which establishes a connection between  $X$  and  $Z$  by means of the natural mappings  $R \rightarrow P \rightarrow X$ ,  $R \rightarrow Q \rightarrow Z$ .

Thus, with the aid of the pair  $(X, \eta)$  we construct a whole class of pairs  $\{(Y, \zeta)\}$ , and this class  $K$  is generated by any one of its pairs  $(Y, \zeta)$ .

Let  $Z, Y \in K$ , and suppose there exists an irreducible mapping  $f : Z \rightarrow Y$ . Among all irreducible mappings  $Z \rightarrow Y$  we choose those that are generated by the system  $\zeta$ . From all the selected mappings we construct the system  $\alpha$ .

Considering together the spaces from the class  $K$  and all mappings from  $\alpha$ , we obtain the category  $\overline{K}$ . Let  $X$  be an arbitrary space of this category. Put  $\widehat{X}(\eta) = \lim\{\beta U\}$ ,  $u = X \setminus F$ ,  $F \in \eta$ .

**Lemma 1.** If  $f : Z \rightarrow X$  is the mapping generated by the system  $\eta$ , then  $\widehat{Z}(f^{-1}(\eta)) = \widehat{X}(\eta)$ , where  $f^{-1}(\eta)$  is the system of all complete inverse images of sets from the system  $\eta$ .

**Lemma 2.** If  $X_1$  and  $X_2$  are spaces of the category  $\overline{K}$ , then  $\widehat{X}_1 = \widehat{X}_2$ .

With the aid of these lemmas it is easy to prove

**Theorem 7.**  $\widehat{X}$  is a projective object in the category  $\overline{K}$ .

By passing to bicomact extensions, all the results obtained can be transferred to arbitrary completely regular spaces.

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## References

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*Note: Figure translations are in progress. See original paper for figures.*

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