

ON THE GENERAL THEORY OF POLYHARMONIC FUNCTIONS

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Abstract

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MATHEMATICS

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ON THE GENERAL THEORY OF POLYHARMONIC FUNCTIONS

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1°. A generalization of the ordinary differential Laplace operator $\Delta^{(p)}U(P)$, which a priori requires the existence of partial derivatives up to order $2p$ inclusive, is the generalized polyharmonic operator $\Delta^{*(p)}U(P)$, introduced by I. I. Privalov and B. K. Pchelin in the work ⁽¹⁾. We shall call $\Delta^{*(p)}U(P)$ the generalized spherical polyharmonic operator of the first kind of order p . It is obvious that the operator $\Delta^{*(p)}U(P)$ does not exist for all functions integrable in the sense of Lebesgue. Thus, for example, consider the function $U(P)$ defined by the conditions: $U(P) = U(x_1, x_2, \dots, x_n)$ at those points $P(x_1, x_2, \dots, x_n)$ of the domain D at which at least one of the coordinates $x_1, x_2, \dots, x_n$ is irrational; $U(P) = 0$ at those points P of the domain D all of whose coordinates $x_1, x_2, \dots, x_n$ are rational. It is easy to prove that for such a function $U(P)$ the operator $\Delta^{*(p)}U(P)$ exists at those points P of the domain D at which at least one of its coordinates is irrational, and does not exist at the remaining points of the domain D . In connection with this there arises the question of a broader definition of the generalized polyharmonic operator.

2°. In the present paper the following notation is adopted: D_{ih} ($i = 1, 2, \dots, p$) is the domain lying inside the domain D and whose boundary is removed from the boundary of the domain D by a distance $> ih$; $\omega(P_i, h)$ and $\sigma(P_i, h)$ are respectively the hypersphere and its surface with center at the point P_i of radius h ; $\omega(h)$ and $\sigma(h)$ are the volume of the hypersphere and the area of its surface; $P = P_p \subset \overline{D}_{ph}$, $P_{p-1} \subset \overline{D}_{(p-1)h}$, $\dots$, $P_1 \subset \overline{D}_h$, $M \subset D$; $C_p^{(k)}$ is the number of combinations of p elements taken k at a time.

3°. Let $U(P)$ be a single-valued function in a certain domain D of n -dimensional space, Lebesgue integrable over every hypersphere and over its surface lying inside the domain D . Consider the expression

$$\delta_h U(P_1) = \frac{1}{\sigma(h)} \int_{\sigma(P_1, h)} U(M) d\sigma - \frac{1}{\omega(h)} \int_{\omega(P_1, h)} U(M) d\omega. \quad (1)$$

This expression represents a function of the point P_1 , Lebesgue integrable in the domain $\overline{D}_h \subset D$. Let the point $P_2 \subset \overline{D}_{2h}$. Repeating operation (1) on the function $\delta_h U(P_1)$, we obtain

$$\delta_h^{(2)} U(P_2) = \delta_h \delta_h U(P_2) = \frac{1}{\sigma(h)} \int_{\sigma(P_2, h)} \left[\frac{1}{\sigma(h)} \int_{\sigma(P_1, h)} U(M) d\sigma - \frac{1}{\omega(h)} \int_{\omega(P_1, h)} U(M) d\omega \right] d\sigma - \frac{1}{\omega(h)} \int_{\omega(P_2, h)} U(M) d\omega$$

Repeating this process p times, we arrive at the expression $\delta_h \delta_h \dots \delta_h U(P) = \delta_h^{(p)} U(P)$, defined at the points $P = P_p \subset \overline{D}_{ph}$. The expressions

$$\overline{\delta^{*(p)} U}(P) = \lim_{h \rightarrow 0} \left[\delta_h^{(p)} U(P) : \frac{h^{2p}}{n^p(n+2)^p} \right],$$

$$\underline{\delta^{*(p)} U}(P) = \lim_{h \rightarrow 0} \left[\delta_h^{(p)} U(P) : \frac{h^{2p}}{n^p(n+2)^p} \right]$$

we shall call, respectively, the upper and lower generalized polyharmonic operators of the second kind of order p for the function $U(P)$. We then arrive at the following definition of a generalized polyharmonic operator of the second kind of order p :

Definition. The function $U(P)$ has at the point P a **generalized polyharmonic operator** $\delta^{*(p)} U(P)$ **of the second kind of order** p , if $\overline{\delta^{*(p)} U}(P) = \underline{\delta^{*(p)} U}(P)$, and we set

$$\delta^{*(p)} U(P) = \lim_{h \rightarrow 0} \left[\delta_h^{(p)} U(P) : \frac{h^{2p}}{n^p(n+2)^p} \right].$$

The operator $\delta^{*(p)} U(P)$ exists, for example, for the function $U(P)$ indicated above. Consequently, it is an operator existing for a broader class of Lebesgue-integrable functions.

To justify this definition, the following is proved.

Theorem 1. Let $U(P)$ be a single-valued function of the point P in some domain D of n -dimensional space, Lebesgue-integrable over every hypersphere and over its surface lying inside the domain D . If for the function $U(P)$ at the points P of the domain D there exists the operator $\Delta^{*(p)} U(P)$, then for the function $U(P)$ at the same points P of the domain D there exists the operator $\delta^{*(p)} U(P)$, equal to $\Delta^{*(p)} U(P)$.

For the proof of this theorem, one first introduces the generalized polyharmonic spherical operator of the first kind of order p , denoted by $\nabla^{*(p)} U(P)$, by means of the equality

$$\nabla^{*(p)}U(P) = \lim_{h \rightarrow 0} \left[\Delta_h^{(p)}U(P) : \frac{h^{2p}}{2^p n^p} \right],$$

where

$$\nabla_h U(P) = \frac{1}{\sigma(h)} \int_{\sigma(P,h)} U(M) d\sigma - U(P),$$

and the existence of this operator and its equality to the operator $\Delta^{*(p)}U(P)$ is proved, provided the latter exist.

3°. In order to arrive at a new definition of a polyharmonic function, Theorems A and B are first proved.

Theorem A. *Let the function $U(P)$ be a polyharmonic function of order p in the domain D . Then, whatever the domain D' , $D' \subset D$, for all points P of this domain the function $\delta_h U(P)$, defined by expression (1), where h is any sufficiently small positive number, is a polyharmonic function of order $p - 1$.*

Theorem B. *If a function $U(P)$, continuous in the domain D , is such that the function $\delta_h U(P)$, defined by expression (1), where h is any sufficiently small positive number, is a polyharmonic function of order $p - 1$, then the function $U(P)$ is a polyharmonic function of order p in the domain D .*

Proceeding from the proved Theorems A and B, we arrive at the following definition of a polyharmonic function:

Definition. A function $U(P)$, continuous in the domain D , is called a **polyharmonic function of order p** in this domain if it satisfies the condition: whatever the domain D' , $D' \subset D$, for all points P of this domain the expression

$$\delta_h U(P) = \frac{1}{\sigma(h)} \int_{\sigma(P,h)} U(M) d\sigma - \frac{1}{\omega(h)} \int_{\omega(P,h)} U(M) d\omega,$$

where h is any sufficiently small positive number, is a polyharmonic function of order $p - 1$.

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1. I. I. Privalov, B. K. Pchelin, *Matem. sborn.*, **2** (44), 745 (1937).

Note: Figure translations are in progress. See original paper for figures.

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