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Abstract

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MATHEMATICS

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THE BEHNKE-STEIN KERNEL AND A CLOSED-FORM SOLUTION OF THE RIEMANN BOUNDARY-VALUE PROBLEM ON A TORUS

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On the torus homeomorphic to the Riemann surface R of the algebraic function $w^2 = (1 - z^2)(1 - k^2 z^2)$, $0 < k < 1$, an explicit analogue of the Cauchy kernel—the Behnke and Stein kernel—is constructed (¹⁻³). With the aid of this kernel, a closed-form solution of the Riemann problem on the surface R is constructed.

1. Points of the Riemann surface R will be written in the form of ordered pairs of numbers (z, w) , connected by the equation $w^2 = (1 - z^2)(1 - k^2 z^2)$. We shall represent the surface R in the form of two sheets located over the z -plane, crosswise glued along the segments $-1/k \leq z \leq 1$ and $1 \leq z \leq 1/k$. By

$$w = \pm \sqrt{(1 - z^2)(1 - k^2 z^2)}$$

we denote the branch of the function

$$\sqrt{(1 - z^2)(1 - k^2 z^2)},$$

single-valued in the cut z -plane, taking at the point $z = 0$ the values ± 1 , respectively. The set of points of the surface R of the form

$$(z, +\sqrt{(1 - z^2)(1 - k^2 z^2)})$$

will be called the upper sheet; the lower sheet of R is defined analogously. The surface R has two points at infinity $(\infty, \pm\infty)$ (one on each sheet) and four branch points $(\pm 1, 0)$, $(\pm 1/k, 0)$. All other points of the surface are regular. As a local parameter σ of a point (z, w) in a neighborhood of a regular point we take $\sigma = z$, in a neighborhood of a point at infinity $\sigma = 1/z$, and in a neighborhood of a branch point $(a, 0)$, $\sigma = \sqrt{z - a}$.

2. Let L be a smooth finite closed contour on R , consisting of regular points and not intersecting the segment (z, w) : $-1 \leq z \leq 1/k$, $\text{Im } z = 0$, bounding a simply connected domain D^+ . By D^- we denote the complement of

$\overline{D}^+$ in R . Let $G(t, v) \neq 0$, $g(t, v)$ be given on L Hölder-continuous functions and a covariant $h(t, v)$. The problems are posed of finding piecewise analytic functions $\Phi^\pm(z, w)$ and covariants $\Psi^\pm(z, w)$ with line of discontinuity L under one of the following boundary conditions on L :

$$\Phi^+(t, v) = G(t, v)\Phi^-(t, v) + g(t, v), \quad (1)$$

$$\Psi^-(t, v) = G(t, v)\Psi^+(t, v) + h(t, v). \quad (2)$$

In works ⁽⁴⁻⁸⁾ problem (1) was studied in a more general setting in sufficient detail from the qualitative point of view on surfaces of arbitrary genus p . In constructing the solution of problem (1) in closed form, the following known results are used ($p = 1$). Let $\varkappa$ be the index of problem (1), and let l and l' be the numbers of solutions of the homogeneous problems (1) and (2), respectively. Then: 1) $l - l' = \varkappa$; 2) if $\varkappa < 0$, then $l = 0$; 3) if $\varkappa > 0$, then $l = \varkappa$; 4) if $\varkappa = 0$, then $0 \leq l \leq 1$ (this estimate is sharp). In view of the fact that the elliptic differential of the first kind dz/w has on R neither zeros nor poles, division of the boundary condition (2) by the covariant $1/v$, $(t, v) \in L$, reduces problem (2) to the equivalent problem (1). Therefore only problem (1) is considered below.

3. Let $(z, w), (\tau, \xi) \in R$. As the Behnke and Stein kernel on R one may take the expression which, at regular points of the surface, has the form

$$\frac{w + \xi}{2\xi} \frac{d\tau}{\tau - z}. \quad (3)$$

This is a function in the variable (z, w) and a differential in the variable (τ, ξ) . The form and behavior of the kernel at singular points are determined by passing in expression (3) to local parameters. We state, without proof, the properties of the kernel (3).

- 1) If the point (z, w) is finite, then, as a function of the variable (τ, ξ) , the kernel (3) is an elliptic (Abelian) differential of the third kind with poles at three points: at the point $(\tau, \xi) = (z, w)$, with residue $d\sigma$, where σ is a local parameter of the point (τ, ξ) in a neighborhood of the point (z, w) ; and at the points $(\tau, \xi) = (\infty, \pm\infty)$, with residues at each of them equal to $-d\sigma/2$, where $\sigma = 1/\tau$.
- 2) If the point (τ, ξ) is finite, then, as a function of the variable (z, w) , the kernel (3) is an algebraic function with poles at three points: at the point (τ, ξ) with residue $(-d\sigma)$, and at the points $(\infty, \pm\infty)$, with expansion near them

$$\frac{w + \xi}{2\xi} \frac{d\tau}{\tau - z} = \pm \frac{k}{2} \frac{d\tau}{\xi} z + \dots$$

Let $\varphi(\tau, \xi)$ be an H -continuous function prescribed on L . The Cauchy-type integral

$$\Phi^\pm(z, w) = \frac{1}{2\pi i} \int_L \frac{w + \xi}{2\xi} \frac{\varphi(\tau, \xi)}{\tau - z} d\tau \quad (4)$$

represents a piecewise meromorphic function on R with line of discontinuity L , for which the Sokhotski formulas hold:

$$\Phi^\pm(t, v) = \pm \frac{1}{2} \varphi(t, v) + \frac{1}{2\pi i} \int_L \frac{v + \zeta}{2\zeta} \frac{\varphi(\tau, \zeta)}{\tau - t} d\tau. \quad (5)$$

The residues of the function (4) at the points $(\infty, \pm\infty)$ are equal to

$$\pm k \int_L \varphi(\tau, \zeta) \frac{d\tau}{4\pi i \zeta}.$$

Let $\psi(\tau, \zeta)$ be an H -continuous covariant prescribed on L . The Cauchy-type integral

$$\Psi^\pm(z, w) = \frac{1}{2\pi i} \int_L \frac{w + \zeta}{2w} \frac{\psi(\tau, \zeta)}{\tau - z} d\tau \quad (6)$$

represents a piecewise meromorphic covariant with line of discontinuity L , for which the Sokhotski formulas analogous to (5) hold. The function (6) has, at the points $(\infty, \pm\infty)$, poles with residues equal to

$$\frac{1}{4\pi i} \int_L \psi(\tau, \zeta) d\tau.$$

Putting $G(t, v) \equiv 1$ in (1) and (2), we obtain the “jump problem.” The solution of problem (1) is given by the integral

$$\Phi^\pm(z, w) = \frac{1}{2\pi i} \int_L \frac{w + \zeta}{2\zeta} \frac{g(\tau, \zeta)}{\tau - z} d\tau + C \quad (7)$$

under the necessary and sufficient solvability condition

$$\int_L g(\tau, \zeta) \frac{d\tau}{\zeta} = 0.$$

The analogous assertion is valid also for problem (2) when $G \equiv 1$.

4. We introduce one more, discontinuous (multi-valued), Behnke-Stein kernel. First we construct canonical cross-sections along the R -cycles A and B . By the cycle A we shall mean a closed curve on R whose projection onto the z -plane coincides with the segment $|z| \leq 1$, $\text{Im } z = 0$. Analogously, the cycle B is projected onto the segment $1 \leq z \leq 1/k$. The period of the differential $d\tau/\zeta$ along the cycle A is equal to $4K$, and along the cycle B it is equal to $2iK'$, where K and K' are complete elliptic integrals (9). As the multivalued (discontinuous) Behnke-Stein kernel one may take the expression

$$\frac{w + \zeta}{2\zeta} \frac{d\tau}{\tau - z} - \frac{1}{4K} \frac{d\tau}{\zeta} \int_A \frac{w + \xi}{2\xi} \frac{d\rho}{\rho - z}, \quad (8)$$

where (z, w) , (τ, ζ) , $(\rho, \xi) \in R$. In contrast to (3), this kernel is analytic in (z, w) . At the points $(\infty, \pm\infty)$ we have, along the line A , a discontinuity with

jump $2\pi i d\tau/4K\zeta$. For the integral with kernel (8), the Sokhotski formulas also hold on L . With the aid of kernel (8), the jump problem is easily solved in the follow-

which is important in solving the homogeneous problem (1) in the formulation. Find a piecewise analytic function on R with lines of discontinuity L and A , satisfying on L the boundary condition

$$\Phi^+(t, v) - \Phi^-(t, v) \equiv g(t, v), \quad (9)$$

and, upon crossing the line A , acquiring an increment in the form of a constant term, whose value is also to be found. The solution (general) of the formulated problem is given by the formula

$$\Phi^\pm(z, w) = \frac{1}{2\pi i} \int_L \frac{w + \zeta}{2\zeta} \frac{g(\tau, \zeta)}{\tau - z} d\tau - \frac{1}{8K\pi i} \int_L g(\tau, \zeta) \frac{d\tau}{\zeta} \int_A \frac{w + \xi}{2\xi} \frac{d\rho}{\rho - z} + C,$$

and the value of the required constant (the difference of the limiting values of the function $\Phi^\pm(z, w)$ on the right and left banks of the cut A) is equal to

$$\frac{1}{4K} \int_L g(\tau, \zeta) \frac{d\tau}{\zeta}.$$

We shall also need the normalized Abelian differential of the third kind $d\omega_{\alpha\beta}(\tau)$ with a pole at the point (α, a) with residue $+1$ and with a pole at the point (β, b) , with residue -1 , regular at the remaining points of the surface R . This differential is constructed by the formula

$$d\omega_{\alpha\beta}(\tau) = \frac{a + \zeta}{2\zeta} \frac{d\tau}{\tau - \alpha} - \frac{b + \zeta}{2\zeta} \frac{d\tau}{\tau - \beta} - \frac{d\tau}{4K\zeta} \int_A \left[\frac{a + \xi}{2\xi} \frac{d\rho}{\rho - \alpha} - \frac{b + \xi}{2\xi} \frac{d\rho}{\rho - \beta} \right]. \quad (10)$$

It has zero A -period, and its B -period is (10)

$$\int_B d\omega_{\alpha\beta}(\tau) = \frac{2\pi i}{4K} \int_{(\beta, b)}^{(\alpha, a)} \frac{d\tau}{\zeta}, \quad (11)$$

where the integration from (β, b) to (α, a) is carried out along a curve not intersecting the lines A and B .

5. The following assertion is true.

For the solvability of the homogeneous problem (1) of index zero it is necessary and sufficient that the numbers m and n , computed by the formulas

$$n = \frac{1}{4K} \operatorname{Re} \left\{ \frac{1}{2\pi i} \int_L \ln G(\tau, \zeta) \frac{d\tau}{\zeta} \right\}, \quad m = -\frac{1}{2K'} \operatorname{Im} \left\{ \frac{1}{2\pi i} \int_L \ln G(\tau, \zeta) \frac{d\tau}{\zeta} \right\}, \quad (12)$$

be integers. If this condition is fulfilled, the general solution of the problem is given by the formula

$$\Phi^\pm(z, w) = C \exp \left\{ \frac{1}{2\pi i} \int_L \frac{w + \zeta}{2\zeta} \frac{\ln G(\tau, \zeta)}{\tau - z} d\tau - \right. \quad (13) \quad \left. - \frac{1}{8K\pi i} \int_L \ln G(\tau, \zeta) \frac{d\tau}{\zeta} \int_A \frac{w + \xi}{2\xi} \frac{d\rho}{\rho - z} - \frac{2\pi i}{8KK'} \operatorname{Im} \left[\frac{1}{2} \right] \right\}$$

This assertion is proved by logarithmizing condition (1) and reducing the problem to the jump problem in the formulation (9). From the conditions which the jump must satisfy upon crossing the line L , conditions (12) are obtained.

6. Consider the homogeneous problem (1) of index one ($g \equiv 0$, $\chi = 1$). In this case the (non-special) problem has one linearly independent solution, which vanishes at some (fixed) point (a, α) of the surface R . Let (β, b) be an arbitrary point taken on the contour L . By $\ln G_\beta(t, v)$ we denote a branch of the function $\ln G(t, v)$ with discontinuity at the point (β, b) . The point (α, a) is the solution of the following Jacobi inversion problem: find a point (α, a) on the cut surface R and integers m and n such that the equality

$$\int_{(0,1)}^{(\alpha,a)} \frac{d\tau}{\zeta} = \int_{(0,1)}^{(\beta,b)} \frac{d\tau}{\zeta} - \frac{1}{2\pi i} \int_L \ln G_\beta(\tau, \zeta) \frac{d\tau}{\zeta} + 4Kn - 2iK'm. \quad (14)$$

The solution of problem (14) is easy to write down explicitly. The numbers m and n are determined uniquely if one requires that the right-hand side of (14) lie in the rectangle with vertices $-3K, K, K + 2iK', -3K + 2iK'$. If in addition it lies to the right of the line $\operatorname{Re} z = K$, then the point $(a, +\sqrt{(1-a^2)(1-k^2a^2)})$ is found from the formula

$$a = \operatorname{sn} \left(\int_{(0,1)}^{(\beta,b)} \frac{d\tau}{\zeta} - \frac{1}{2\pi i} \int_L \ln G_\beta(\tau, \zeta) \frac{d\tau}{\zeta} \right). \quad (15)$$

In the opposite case the point $(a, -\sqrt{(1-a^2)(1-k^2a^2)})$ is found from the formula

$$a = \operatorname{sn} \left(2K - \int_{(0,1)}^{(\beta,b)} \frac{d\tau}{\zeta} + \frac{1}{2\pi i} \int_L \ln G_\beta(\tau, \zeta) \frac{d\tau}{\zeta} \right), \quad (16)$$

where $\text{sn}(x)$ is the Jacobi elliptic function ⁹.

The general solution of the homogeneous problem $\varkappa = 1$ has the form:

$$\Phi^\pm(z, w) = C \exp \left\{ \frac{1}{2\pi i} \int_L \frac{w + \zeta}{2\zeta} \frac{\ln G_\beta(\tau, \zeta)}{\tau - z} d\tau - \frac{1}{8K\pi i} \int_L \ln G_\beta(\tau, \zeta) \frac{d\tau}{\zeta} \int_A \frac{w + \xi}{2\xi} \frac{d\rho}{\rho - z} + \frac{2\pi i m}{4K} \int_{(0,1)}^{(z,w)} \frac{d\tau}{\zeta} + \int_{(0,1)}^{(z,w)} d\omega_{a\beta}(\tau) \right\}, \quad (17)$$

where all symbols have been defined above.

7. The solution of the homogeneous problem (1) of arbitrary index is easily reduced to the problem of index 1 considered in the preceding paragraph. Indeed, the problem

$$\Phi_0^+ = \sqrt[\varkappa]{G} \Phi_0^-$$

has index 1; hence it can be solved by formula (17). But then $\{\Phi_0^\pm\}^\varkappa$ will be a particular solution of the homogeneous problem (1) of index $\varkappa$, having at the point (a, a) a zero of order $\varkappa$. Finding the general solution is now reduced to constructing algebraic functions of the equation $w^2 = (1 - z^2)(1 - k^2 z^2)$ having at the point (a, a) poles of orders $2, 3, \dots, \varkappa$. But such functions are easily constructed explicitly.

At present we can give a closed-form solution of the nonhomogeneous problem (1) only in the case when the corresponding homogeneous problem is nontrivially solvable. For this one takes a particular solution $\Phi_0^\pm(z, w)$ of the corresponding homogeneous problem, and problem (1) is reduced to a jump problem. For $\varkappa = 0$ the latter is solved with the aid of the kernel (3). For $\varkappa > 0$, the resulting jump problem (now posed in the class of meromorphic functions, because of the presence of zeros of the function $\Phi_0^\pm(z, w)$) can be solved with the aid of the kernel

$$\frac{w + \zeta}{2\zeta} \frac{d\tau}{\tau - z} - \frac{d\tau}{2\varkappa\xi} \sum_{n=1}^{\varkappa} \frac{w + a_n}{a_n - z},$$

where the points (a_n, a_n) are zeros of the function $\Phi_0^\pm(z, w)$, which for $\varkappa > 1$ can be made simple, regular, and not lying on L .

On the basis of the given solution of the Riemann problem one can solve the singular integral equation with the Behnke-Stein kernel.

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