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Abstract

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MATHEMATICS

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SOME DIRECT AND INVERSE THEOREMS IN THE THEORY OF BEST APPROXIMA- TIONS OF FUNCTIONS BY ALGEBRAIC POLYNOMIALS

(Presented by Academician S. L. Sobolev on 2 XII 1968)

As is known (see, for example, ⁽⁷⁾, p. 355, or ⁽⁴⁾, pp. 165-167), the inverse theorems in the theory of best approximations of functions defined on a finite interval by means of algebraic polynomials, in contrast with the periodic case, are less complete and make it possible to judge the differential properties of the functions under consideration not on the whole interval of definition, but only on intervals lying entirely inside it. In the present paper, in other terms, some direct and inverse theorems are obtained; moreover, in the inverse theorems, from the order of decrease of the best approximation, structural properties of the functions are established already on the whole interval of definition. In particular, structural properties are established for the class of functions satisfying the condition

$$E_n(f) \leq \frac{M}{n^s} \varphi\left(\frac{1}{n}\right),$$

where $s \geq 0$ is an even number; M is a constant independent of n ; $E_n(f)$ denotes the best approximation of the function $f(x)$ in the norm of the class $C[-1, 1]$ by algebraic polynomials of degree not exceeding $n-1$; $\varphi(\delta)$ is a given nondecreasing function. The question of absolute convergence of Fourier-Legendre series is also considered.

For each function $f(x) \in C[-1, 1]$, set

$$\{f(x)\}_h = \frac{1}{\pi} \int_0^\pi f(x \cos h + \sqrt{1-x^2} \sin h \cos \theta) d\theta, \quad h > 0,$$

$$\omega_f(\delta) = \sup_{h \leq \delta} \|f - \{f\}_h\|_{C[-1,1]},$$

$$\Omega(f; \delta) = \sup_{C>0} \frac{\omega_f(C\delta)}{(1+C)^2}.$$

By $C, C_1, C_2, \dots$ we shall denote constants depending on the indicated parameters.

The following theorem holds, which is an analogue of the well-known theorem of D. Jackson.

Theorem 1. If the Legendre operator can be applied to a function $f(x) \in C[-1, 1]$ $p \geq 0$ times,

$$D \equiv \frac{d}{dx}(1-x^2)\frac{d}{dx},$$

and the result of the application, i.e. the function $D^p f(x) \in C[-1, 1]$, then for every natural n the inequality

$$E_n(f) \leq \frac{C_1(p)}{n^{2p}} \Omega\left(D^p f; \frac{1}{n}\right).$$

holds.

We now formulate the inverse theorems.

Theorem 2. For any function $f(x) \in C[-1, 1]$, for every natural n the inequality

$$\Omega\left(f; \frac{1}{n}\right) \leq \frac{C_2}{n^2} \sum_{k=1}^n k E_k(f).$$

Theorem 3. Let $f(x) \in C[-1, 1]$, and suppose that for some natural number p the series

$$\sum_{n=1}^{\infty} n^{2p-1} E_n(f)$$

converges.

Then the Legendre operator D is applicable to the function $f(x)$ p times; the result of its application, i.e. the function $D^p f(x)$, belongs to $C[-1, 1]$, and the inequalities

$$1) \quad E_n(D^p f) \leq C_3(p) \left\{ n^{2p} E_n(f) + \sum_{k=n+1}^{\infty} k^{2p-1} E_k(f) \right\};$$

$$2) \quad \Omega \left(D^p f; \frac{1}{n} \right) \leq C_4(p) \left\{ \frac{1}{n} \sum_{k=1}^n k^{2p+1} E_k(f) + \sum_{k=n+1}^{\infty} k^{2p-1} E_k(f) \right\}.$$

The proofs of Theorems 2 and 3 are based on the following principal lemmas.

Lemma 1. If $P_n(x)$ is an algebraic polynomial of degree n , then

$$\|DP_n\|_{C[-1,1]} \leq 2n^2 \|P_n\|_{C[-1,1]}.$$

This inequality, which is an analogue of the well-known inequality of S. N. Bernstein for derivatives of trigonometric polynomials, is due to G. G. Kushnirenko ⁽³⁾. It can be obtained (but without the exact constant) from an inequality for harmonic polynomials proved earlier by A. L. Shaginyan ⁽⁹⁾.

Lemma 2. For any algebraic polynomial $P_n(x)$ the inequality

$$\|P_n - \{P_n\}_h\|_{C[-1,1]} \leq h^2 \|DP_n\|_{C[-1,1]}$$

holds.

This inequality is valid for any function $f(x) \in C[-1, 1]$, if this function admits application of the Legendre operator D , and the result of the application, i.e. the function $Df(x)$, is bounded on the interval $[-1, 1]$.

From Theorems 1, 2, and 3, as a consequence, there follows

Theorem 4. Let the function $\varphi(\delta)$ satisfy the $(\mathfrak{L}_2)$ -condition of S. M. Lozinskii ⁽¹⁾, and let $p \geq 0$ be an integer. Then, in order that

$$E_n(f) = O \left(n^{-2p} \varphi \left(\frac{1}{n} \right) \right),$$

it is necessary and sufficient that the function $f(x)$ admit p -fold application of the Legendre operator D , the result of the application, i.e. the function $D^p f(x)$, belong to $C[-1, 1]$, and that

$$\Omega(D^p f; \delta) = O(\varphi(\delta)).$$

For $p = 0$ it is sufficient to require that the function $\varphi(\delta)$ satisfy the (N^α) -condition of S. B. Stechkin ⁽⁵⁾, where $0 < \alpha < 2$.

It should be noted that the question of the structural properties of the class of functions satisfying the condition

$$E_n^{(2)}(f) \leq M/n^{r+\beta},$$

where $E_n^{(2)}(f)$ is the best approximation of the function $f(x)$ on $[-1, 1]$ by algebraic polynomials of degree not exceeding n in the L_2 metric, was considered by G. V. Zhidkov ⁽²⁾.

Theorem 4 in the case $p = 0$ admits the following strengthening.

Theorem 5. Let the function $\varphi(\delta)$ satisfy the (N^α) -condition of S. B. Stechkin, where $0 < \alpha < 2$. Then the relation

$$E_n(f) \sim \varphi\left(\frac{1}{n}\right)$$

and

$$\Omega(f; \delta) \sim \varphi(\delta) *$$

are equivalent.

Theorems 2-5 are analogous to the corresponding theorems of S. B. Stechkin ⁽⁵⁾, A. F. Timan and M. F. Timan ⁽⁸⁾.

We note that in Theorems 4 and 5 one may take, for example, the function δ^α , where $0 < \alpha < 2$, as the function $\varphi(\delta)$.

Let $\widetilde{H}^{(r,\alpha)}[-1, 1]$ be the class of functions in $C[-1, 1]$ having an r -th derivative satisfying a Lipschitz condition of order α , and let $\widetilde{H}^{(p,\beta)}[-1, 1]$ ($\widetilde{H}^{(p)}\widetilde{W}[-1, 1]$) be the class of functions in $C[-1, 1]$ admitting p -fold application of the Legendre operator D , with $D^p f(x) \in C[-1, 1]$ and

$$\Omega(D^p f; \delta) = O(\delta^\beta) \quad (\Omega(D^p f; \delta) = O(\delta^2 |\ln \delta|)).$$

For $r = p = 0$, it is convenient to denote the corresponding classes by $H^{(\alpha)}$, $\widetilde{H}^{(\beta)}$, and $\widetilde{W}$.

From the well-known theorems of D. Jackson and S. N. Bernstein on best approximations of nonperiodic functions, and from Theorem 4 (for $\varphi(\delta) = \delta^\beta$, $0 < \beta < 2$), 2 and 3, it follows that

Theorem 6. The following inclusion holds

$$H^{(r,\alpha)} \subset \begin{cases} H^{(r/2,\alpha)}, & \text{for even } r \text{ and } 0 < \alpha \leq 1, \\ \widetilde{H}^{([r/2],1+\alpha)}, & \text{for odd } r \text{ and } 0 < \alpha < 1, \\ \widetilde{H}^{([r/2])}\widetilde{W}, & \text{for even } r \text{ and } \alpha = 1. \end{cases}$$

Let $f(x) \in C[-1, 1]$ and

$$\sum_{n=0}^{\infty} a_n P_n(x), \quad \text{where} \quad a_n = \frac{2n+1}{2} \int_{-1}^1 f(x) P_n(x) dx,$$

be its Fourier-Legendre series.

P. K. Suetin ⁽⁶⁾ proved that every function $f(x)$ satisfying on $[-1, 1]$ a Lipschitz condition of order $\alpha > 1/2$ expands in a Fourier-Legendre series converging uniformly on this segment (he also established an estimate for the deviation of the function from its partial Fourier-Legendre sum in the class $H^{(r,\alpha)}$). From a result of G. V. Zhidkov ⁽²⁾ and from Theorem 6 it follows that this same condition is sufficient for convergence of the series

$$\sum_{n=1}^{\infty} n^{-1/2} |a_n|,$$

which, in turn, ensures the absolute convergence almost everywhere on $[-1, 1]$ of the Fourier-Legendre series of the function $f(x)$.

P. K. Suetin obtained his results using a strengthening, due to A. F. Timan, of D. Jackson's theorem. We note that these results follow from the more precise inequality

$$\left\| f(x) - \sum_{k=0}^n a_k P_k(x) \right\|_{C[-1,1]} \leq C_5 \sqrt{n} E_n(f),$$

which may be regarded as known.

Theorem 7. If $f(x) \in \widetilde{H}^{(\beta)}[-1, 1]$, $\beta > 1$, then

$$\sum_{n=1}^{\infty} |a_n| < \infty,$$

i.e. the Fourier-Legendre series of the function $f(x)$ converges absolutely (and uniformly) on $[-1, 1]$.

* The symbol $\varphi(t) \sim \psi(t)$ is understood as follows: there exist constants $A > 0$ and $B > 0$

such that, for all t for which φ and ψ are defined,

$$A\psi(t) \leq \varphi(t) \leq B\psi(t).$$

Hence, on the basis of Theorem 6, it follows that

Corollary 1. If $f(x) \in H^{(1,\varepsilon)}[-1, 1]$, $\varepsilon > 0$, then the Fourier-Legendre series of the function $f(x)$ converges absolutely on $[-1, 1]$.

Remark. The inequalities formulated in Lemmas 1 and 2, and, consequently, Theorems 2 and 3, are also valid in the metric of the space L_2 .

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REFERENCES

- ¹ N. K. Bari, S. V. Stechkin, *Tr. Moskov. matem. obshch.*, **5**, 483 (1956).
- ² G. V. Zhukov, *DAN*, **169**, No. 5 (1966).
- ³ G. G. Kushnirenko, *Nauchn. dokl. vyssh. shkoly, ser. fiz.-matem. nauk*, No. 4 (1958).
- ⁴ I. P. Natanson, *Constructive Function Theory*, Moscow-Leningrad, 1949.
- ⁵ S. B. Stechkin, *Izv. AN SSSR, ser. matem.*, **15**, No. 3 (1951).
- ⁶ P. K. Suetin, *DAN*, **158**, No. 6 (1964).
- ⁷ A. F. Timan, *Theory of Approximation of Functions of a Real Variable*, Moscow, 1960.
- ⁸ A. F. Timan, M. F. Timan, *DAN*, **71**, No. 1 (1950).
- ⁹ A. L. Shaginyan, *DAN*, **90**, No. 2 (1953).

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