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Abstract

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THEORY OF ELASTICITY

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ON THE INTEGRATION OF A SYSTEM OF EQUATIONS OF ELASTIC EQUILIBRIUM OF A PLATE

The problem of studying the elastic equilibrium of a plate, following the shell theory developed in ^(1,2), leads to the following elliptic system of equations of the 12th order (approximation of order $N = 1$):

$$\begin{aligned}\mu\Delta u_1 + (\lambda + \mu)\partial\theta_1/\partial x + \lambda\partial v/\partial x &= X_1, \\ \mu\Delta u_2 + (\lambda + \mu)\partial\theta_1/\partial y + \lambda\partial v/\partial y &= X_2 \quad (\theta_1 = \partial u_1/\partial x + \partial u_2/\partial y); \\ \mu\Delta v - 12((\lambda + 2\mu)v + \lambda\theta_1) &= X_3;\end{aligned}\quad (1)$$

$$\begin{aligned}\mu\Delta v_1 + (\lambda + \mu)\partial\theta_2/\partial x - 12\mu(\partial u/\partial x + v_1) &= Y_1, \\ \mu\Delta v_2 + (\lambda + \mu)\partial\theta_2/\partial y - 12\mu(\partial u/\partial y + v_2) &= Y_2, \\ (\theta_2 = \partial v_1/\partial x + \partial v_2/\partial y); \\ \mu\Delta u + \mu\theta_2 &= Y_3,\end{aligned}\quad (2)$$

where $X_1, X_2, X_3, Y_1, Y_2, Y_3$ are prescribed functions; x and y are dimensionless (Descartes) coordinates; the metric quadratic form on the middle plane has the form $ds^2 = 4h^2(dx^2 + dy^2)$, where $2h$ is the thickness of the plate (h is constant).

The functions entering into (1) and (2) have the following kinematic meaning: $u + tv$ is the normal deflection, $u_1 + tv_1$ and $u_2 + tv_2$ are tangential displacements on the plane parallel to the middle plane and at a distance $2ht$ from it, $-1/2 \leq t \leq 1/2$; u is the deflection of the middle plane, v is the elongation of transverse fibers. In the classical theory of shells, based on the Kirchhoff-Love hypothesis, the quantity v is neglected, taking $v = 0$.

For $X_i = Y_i = 0$ ($i = 1, 2, 3$), systems (1) and (2) are integrated in explicit form ⁽¹⁻³⁾

$$v = \chi - \frac{\lambda}{\lambda + 2\mu} \Delta\varphi; \quad (3)$$

$$u_1 = -\frac{\lambda}{48(\lambda + \mu)} \frac{\partial \chi}{\partial x} + \frac{\partial \varphi}{\partial x} - \frac{\partial \varphi^*}{\partial y}, \quad u_2 = -\frac{\lambda}{48(\lambda + \mu)} \frac{\partial \chi}{\partial y} + \frac{\partial \varphi}{\partial y} + \frac{\partial \varphi^*}{\partial x}; \quad (4)$$

$$u = -\omega + B_0 \Delta \omega \quad (B_0 = (\lambda + 2\mu)/12\mu); \quad (5)$$

$$v_1 = \partial \omega / \partial x - \partial \psi / \partial y, \quad v_2 = \partial \omega / \partial y + \partial \psi / \partial x, \quad (6)$$

where χ and ψ are arbitrary solutions of the equations*

$$\Delta \chi - k^2 \chi = 0, \quad \Delta \psi - m^2 \psi = 0 \quad (k^2 = 48(\lambda + \mu)/(\lambda + 2\mu), \quad m^2 = 12), \quad (7)$$

$\varphi, \varphi^*, \omega$ are biharmonic functions, with φ and ω arbitrary, while φ^* is expressed in terms of φ ; if, for example, φ is written in the form

$$\varphi = \frac{\lambda + 2\mu}{2(3\lambda + 2\mu)} \left(z \bar{f} + \bar{z} f - \frac{1}{2}(f_0 + \bar{f}_0) \right), \quad (8)$$

then

$$\varphi^* = i \frac{2(\lambda + \mu)}{3\lambda + 3\mu} (z \bar{f} - \bar{z} f). \quad (9)$$

Here f and f_0 are arbitrary analytic functions of $z = x + iy$.

* The equation $\Delta \psi - m^2 \psi = 0$ first occurs in E. Reissner ⁽⁶⁾, but there $m^2 = 10$. The reason for this discrepancy is indicated in ⁽³⁾.

Let $\widetilde{X}_i, \widetilde{Y}_i$ ($i = 1, 2, 3$) be functions satisfying the equations

$$\Delta \Delta \widetilde{X}_i = X_i, \quad \Delta \Delta \widetilde{Y}_i = Y_i \quad (i = 1, 2, 3). \quad (10)$$

Differentiating the first equation of system (1) with respect to x , the second with respect to y , and then adding the results, by virtue of (10) we shall have

$$(\lambda + 2\mu) \Delta \theta_1 + \lambda \Delta v = \Delta \Delta \nabla^\alpha \widetilde{X}_\alpha, \quad (11)$$

where ∇^α denotes the symbol of the contravariant derivative.

Equation (11) is satisfied by the function

$$(\lambda + 2\mu) \theta_1 + \lambda v = \Delta \nabla^\alpha \widetilde{X}_\alpha. \quad (12)$$

Hence we have

$$\theta_1 = -\frac{\lambda}{\lambda + 2\mu}v + \frac{1}{\lambda + 2\mu}\Delta\nabla^\alpha\tilde{X}_\alpha. \quad (13)$$

Substituting this into the third equation of system (1), we obtain

$$(\Delta - k^2)v = \Delta\left(\frac{1}{\mu}\Delta\tilde{X}_3 + \frac{12\lambda}{\mu(\lambda + 2\mu)}\nabla^\alpha\tilde{X}_\alpha\right). \quad (14)$$

Let $L_k(g)$ be an operator giving some particular solution of the equation $(\Delta - k^2)w = g$ ($k^2 = \text{const}$). The operator L_k can always be chosen in such a form that the permutation formulas (in the case of Cartesian coordinates) $L_k(\nabla_\alpha g) = \nabla_\alpha L_k(g)$ ($\alpha = 1, 2$) hold.

Representing v by the formula

$$v = \Delta L_k\left(\frac{1}{\mu}\Delta\tilde{X}_3 + \frac{12\lambda}{\mu(\lambda + 2\mu)}\nabla^\alpha\tilde{X}_\alpha\right) \quad (15)$$

and inserting this expression into (13), we obtain

$$\theta_1 = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = \Delta\left(\frac{\lambda}{\lambda + 2\mu}\nabla^\alpha\tilde{X}_\alpha - \frac{\lambda}{\mu(\lambda + 2\mu)}L_k\left(\Delta\tilde{X}_3 + \frac{12\lambda}{\lambda + 2\mu}\nabla^\alpha\tilde{X}_\alpha\right)\right). \quad (16)$$

Differentiating now the first equation of system (1) with respect to y , the second with respect to x , and then subtracting, we shall have

$$\mu\Delta\theta_1^* = \Delta\Delta c^{\alpha\beta}\nabla_\beta\tilde{X}_\alpha, \quad \theta_1^* = \partial u_1/\partial y - \partial u_2/\partial x, \quad (17)$$

where $c^{\alpha\beta}$ is the contravariant discriminant tensor (4). From (17) we have

$$\theta_1^* = \frac{\partial u_1}{\partial y} - \frac{\partial u_2}{\partial x} = \nabla\left(\frac{1}{\mu}c^{\alpha\beta}\Delta_\beta\tilde{X}_\alpha\right). \quad (18)$$

Equalities (16) and (18) can be written in the following complex form:

$$\begin{aligned} 2\partial(u_1 - iu_2)/\partial\bar{z} &= \theta_1 + i\theta_1^* \\ &= \Delta\left(\frac{1}{\lambda + 2\mu}\nabla^\alpha\tilde{X}_\alpha - \frac{\lambda}{\mu(\lambda + 2\mu)}L_k\left(\Delta\tilde{X}_3 + \frac{12\lambda}{\lambda + 2\mu}\nabla^\alpha\tilde{X}_\alpha\right) + \frac{i}{\mu}c^{\alpha\beta}\nabla_\beta\tilde{X}_\alpha\right). \end{aligned} \quad (19)$$

Since $\Delta = 4\partial^2/\partial z\bar{z}$, from (19) we have

$$u_1 + iu_2 = 2\frac{\partial}{\partial z} \left(\frac{1}{\lambda + 2\mu} \nabla^\alpha \tilde{X}_\alpha - \frac{\lambda}{\mu(\lambda + 2\mu)} L_k \left(\Delta \tilde{X}_3 + \frac{12\lambda}{\lambda + 2\mu} \nabla^\alpha \tilde{X}_\alpha \right) - \frac{i}{\mu} c^{\alpha\beta} \nabla_\beta \tilde{X}_\alpha \right). \quad (20)$$

Formulas (15) and (20) give particular solutions of the nonhomogeneous system (1).

Let X_1, X_2 be the covariant components of a vector, and X_3 a scalar.

Representing now the Laplace operator in the form $\Delta = \nabla^\alpha \nabla_\alpha = \nabla^2$, formulas (15) and (20) can be written in tensor form:

$$v = \nabla^2 L_k \left(\frac{1}{\mu} \nabla^2 \tilde{X}_3 + \frac{12\lambda}{\mu(\lambda + 2\mu)} \nabla^\alpha \tilde{X}_\alpha \right), \quad (21)$$

$$u_\alpha = \nabla^\beta \left(a_{\alpha\beta} \left(\frac{1}{\lambda + 2\mu} \nabla^\gamma \tilde{X}_\alpha - \frac{\lambda^\gamma}{\mu(\lambda + 2\mu)} L_k \left(\nabla^2 \tilde{X}_3 + \frac{12\lambda}{\lambda + 2\mu} \nabla^\gamma \tilde{X}_\gamma \right) \right) + \frac{1}{\mu} c_{\alpha\beta} c^{\gamma\nu} \nabla_\nu \tilde{X}_\gamma \right), \quad (22)$$

where $a_{\alpha\beta}$ is the covariant metric tensor, and $c_{\alpha\beta}$ is the covariant discriminant tensor.

With respect to arbitrarily chosen curvilinear coordinates x^1, x^2 , equation (1) can be written in the form

$$\mu \nabla^\alpha \nabla_\alpha u_\beta + (\lambda + \mu) \nabla_\beta \nabla^\alpha u_\alpha + \lambda \nabla_\beta v = X_\beta \quad (\beta = 1, 2),$$

$$\mu \nabla^2 v - 12((\lambda + 2\mu)v + \lambda \nabla^\alpha u_\alpha) = X_3. \quad (23)$$

Let us now turn to equations (2). Differentiating the first equation with respect to x , the second with respect to y , and then adding the results, by virtue of the third equation we obtain

$$(\lambda + 2\mu) \Delta \theta_2 = 12 \Delta \Delta \tilde{Y}_3 + \Delta \Delta \nabla^\alpha \tilde{Y}_\alpha.$$

Hence we have

$$\theta_2 = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = \Delta \left(\frac{12}{\lambda + 2\mu} \tilde{Y}_3 + \frac{1}{\lambda + 2\mu} \nabla^\alpha \tilde{Y}_\alpha \right). \quad (24)$$

Differentiating the first equation of system (2) with respect to y , the second with respect to x , and then subtracting, we shall have

$$\mu(\Delta\theta_2^* - m^2\theta_2^*) = \Delta\Delta c^{\alpha\beta}\nabla_\beta\tilde{Y}_\alpha \quad (\theta_2^* = \partial v_1/\partial y - \partial v_2/\partial x; \quad m^2 = 12).$$

Hence we have

$$\theta_2^* = \frac{\partial v_1}{\partial y} - \frac{\partial v_2}{\partial x} = \Delta\Delta \left(\frac{1}{\mu} L_m (c^{\alpha\beta}\nabla_\beta\tilde{Y}_\alpha) \right). \quad (25)$$

Equalities (24) and (25) can be written in complex form as follows:

$$2\frac{\partial(v_1 - iv_2)}{\partial\bar{z}} = \theta_2 + i\theta_2^* = \Delta \left(\frac{12}{\lambda + 2\mu}\tilde{Y}_3 + \frac{1}{\lambda + 2\mu}\nabla^\alpha\tilde{Y}_\alpha + \frac{i}{\mu}\Delta L_m (c^{\alpha\beta}\nabla_\beta\tilde{Y}_\alpha) \right),$$

i.e., we may assume that

$$v_1 + iv_2 = 2\frac{\partial}{\partial\bar{z}} \left(\frac{12}{\lambda + 2\mu}\tilde{Y}_3 + \frac{1}{\lambda + 2\mu}\nabla^\alpha\tilde{Y}_\alpha - \frac{i}{\mu}\Delta L_m (c^{\alpha\beta}\nabla_\beta\tilde{Y}_\alpha) \right) \quad (26)$$

or, in tensor form,

$$v_\alpha = \nabla^\beta \left(a_{\alpha\beta} \left(\frac{12}{\lambda + 2\mu}\tilde{Y}_3 + \frac{1}{\lambda + 2\mu}\nabla^\gamma\tilde{Y}_\gamma \right) + \frac{1}{\mu}c_{\alpha\beta}\nabla^2 L_m (c^{\gamma\nu}\nabla_\nu\tilde{Y}_\gamma) \right) \quad (\alpha = 1, 2). \quad (27)$$

Now, by virtue of the third equation of system (2) and equality (24), for u we obtain the formula

$$u = \frac{1}{\mu}\nabla^2\tilde{Y}_3 - \frac{12}{\lambda + 2\mu}Y_3 - \frac{1}{\lambda + 2\mu}\nabla^\alpha\tilde{Y}_\alpha. \quad (28)$$

Formulas (27) and (28) give particular solutions of the system of equations (2). This system in tensor form can be written as

$$\mu\nabla^\alpha\nabla_\alpha v_\beta + (\lambda + \mu)\nabla_\beta\nabla^\alpha v_\alpha - 12\mu(\nabla_\beta u + v_\beta) = Y_\beta \quad (\beta = 1, 2),$$

$$\mu\nabla^2 u + \mu\nabla^\alpha v_\alpha = Y_3. \quad (29)$$

Now let us return to the question of constructing particular solutions of the equations $\Delta\Delta w = g$ and $(\Delta - k^2)w = g$ ($k^2 = \text{const}$). Their particular solutions can be constructed, respectively, by the formulas

$$w = P_0(g) = \frac{1}{8\pi} \iint_E |z - \xi|^2 \ln |z - \xi| g(\xi, \eta) d\xi d\eta; \quad (30)$$

$$w = L_k(g) = \frac{1}{2\pi} \iint_E K_0(k, r) g(\xi, \eta) d\xi d\eta, \quad r = |z - \xi|, \quad (31)$$

where K_0 is the modified Bessel function of the second kind⁵. If g has continuous partial derivatives on the plane E , then it is easy to verify the validity of the following commutation formulas (with respect to Cartesian coordinates):

$$P_0(\nabla_\alpha g) = \nabla_\alpha P_0(g), \quad L_k(\nabla_\alpha g) = \nabla_\alpha L_k(g) \quad (\alpha = 1, 2).$$

Lemma. Let g_1, g_2 be the covariant components of a vector. Suppose that they are continuously differentiable in E and vanish outside a circle of sufficiently large radius. Then there exist scalars u, v , by means of which g_1, g_2 are expressed in the form $g_\alpha = \nabla^\beta (a_{\alpha\beta} u + c_{\beta\alpha} v)$ ($\alpha = 1, 2$).

As u and v one may take scalars satisfying the equations: $\nabla^2 u = \nabla^\alpha g_\alpha$, $\nabla^2 v = c^{\alpha\beta} \nabla_\beta g_\alpha$; they may be expressed by the formulas

$$u = \frac{1}{2\pi} \iint_E \ln |z - \xi| \nabla^\alpha g_\alpha d\xi d\eta, \quad v = \frac{1}{2\pi} \iint_E \ln |z - \xi| c^{\alpha\beta} \nabla_\beta g_\alpha d\xi d\eta.$$

On the basis of this lemma it is easily proved that particular solutions of the equations $\nabla^2 w_\alpha = g_\alpha$, $(\nabla^2 - k^2)w_\alpha = g_\alpha$ can be constructed, respectively, by the formulas

$$w_\alpha = \nabla^\beta (a_{\alpha\beta} P_0(u) + c_{\beta\alpha} P_0(v)), \quad w_\alpha = \nabla^\beta (a_{\alpha\beta} L_k(u) + c_{\beta\alpha} L_k(v)).$$

Obviously, w_1, w_2 are the components of a covariant vector.

Let g satisfy the equation $\nabla^2 g - p^2 g = 0$, $p^2 = \text{const} \neq 0$. Then $P_0(g)$ and $L_k(g)$ can be expressed by the formulas

$$P_0(g) = \frac{1}{p^4} g, \quad L_k(g) = \begin{cases} \frac{1}{p^2 - k^2} g, & \text{if } p^2 \neq k^2, \\ r \frac{\partial g}{\partial r}, & \text{if } p^2 = k^2. \end{cases}$$

If g satisfies the equation $\nabla^{2n} g = 0$, then

$$L_p(g) = -\frac{1}{p^4} \left(g + \frac{1}{p^2} \nabla^2 g + \dots + \frac{1}{p^{2n-2}} \nabla^{2n-2} g \right),$$

$$g = \sum_{k=0}^{n-1} (z^k \bar{f}_k + \bar{z}^k f_k),$$

$$w = P_0(g) = \frac{1}{16} \sum_{k=0}^{n-1} \frac{1}{(k+1)(k+2)} (z^{k+2} \bar{f}_k + \bar{z}^{k+2} f_k),$$

where f_k are arbitrary analytic functions of z .

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Note: Figure translations are in progress. See original paper for figures.

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