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Abstract

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MATHEMATICS

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A PRIORI ESTIMATES AND HYPOELLIPTIC OPERATORS WITH MULTIPLE CHARACTERISTICS

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In the paper (¹) L. Hörmander, for a second-order differential operator of the form

$$Pu = \sum_{j=1}^N X_j^2 u + iX_0 u + \gamma(x)u, \quad (1)$$

where $X_j(x, D)$ ($j = 0, 1, \dots, N$) are first-order differential operators with infinitely differentiable coefficients, obtained sufficient conditions for the existence of estimates

$$\sum_{j=1}^N \|X_j u\|_{(\varepsilon)}^2 + \|u\|_{(2\varepsilon)}^2 \leq C(K) \{\|Pu\|_{(0)}^2 + \|u\|_{(0)}^2\}, \quad u \in C_0^\infty(K), \quad (2)$$

where K is any compact set in the domain $\Omega \subset R^n$ and the constant $\varepsilon = \varepsilon(K) > 0$ (see (²)); it can be shown that Hörmander's conditions are also necessary in the class of operators of the form (1) for the existence of the estimate (2) with some $\varepsilon(K) > 0$ ($D_j = -i\partial/\partial x_j$, $j = 1, \dots, n$; $i^2 = -1$). In the paper (²) a simple proof of Hörmander's theorem was obtained. Using the same method, we shall consider the general case of second-order differential operators with nonnegative characteristic form, namely:

$$Lu \equiv \sum_{k,j=1}^n a^{kj}(x) D_k D_j u + iX_0(x, D)u + \gamma(x)u, \quad (3)$$

where $a^{kj}(x)\xi_k\xi_j \geq 0$ for any point $(x, \xi) \in \Omega \times S^{n-1}$.

The question of global smoothness of generalized solutions of boundary-value problems for equation (3) was considered in the papers (³⁻⁶). We shall give

sufficient conditions for local smoothness of generalized solutions of equation (3).

In the domain $\Omega \subset R^n$ consider the system of differential operators

$$\{L^{0(j)}, j = 1, \dots, n; \quad L_{(j)}^0, j = 1, \dots, n; \quad X_0 + L_{(j)}^{0(j)}\},$$

where

$$L^{0(j)}(x, D) \equiv \sum_k a^{kj}(x) D_k \quad (j = 1, \dots, n);$$

$$L_{(s)}^0(x, D) \equiv \sum_{k,j} \frac{\partial a^{kj}(x)}{\partial x_s} D_k D_j \quad (s = 1, \dots, n); \quad L_{(j)}^{0(j)} \equiv \sum_{k,j} \frac{\partial a^{kj}}{\partial x_j} D_k.$$

For any multi-index $I = (\alpha_1, \dots, \alpha_k)$ ($k \geq 1$), where $\alpha_s = 0, 1, \dots, 2n$, for any $s = 1, \dots, k$ put: $|I| = \sum_{s=1}^k \lambda_s$, where $\lambda_s = 2$ if $\alpha_s = 0$, and $\lambda_s = 1$ if $\alpha_s = 1, \dots, 2n$, and let

$$A_I(x, D) \equiv \text{ad } A_{\alpha_1} \dots \text{ad } A_{\alpha_{k-1}} A_{\alpha_k},$$

where $A_s(x, D) \equiv L^{0(s)}(x, D)$ ($s = 1, \dots, n$); $A_{s+n}(x, D) \equiv L_{(s)}^0(x, D)$ ($s = 1, \dots, n$); $A_0(x, D) \equiv X_0 + L_{(j)}^{0(j)}$ and $\text{ad } AB = AB - BA$ for any pseudodifferential operators A, B (see, for example, (7)).

Definition. The system of operators $(A_0, \dots, A_{2n})$ has at the point $x_0 \in \Omega$ rank $R_{(A_0, \dots, A_{2n})}(x_0) = k$, if

$$\sum_{|I| \leq k-1} |A_I^0(x_0, \xi)| = 0 \quad \text{for some } \xi \in S^{n-1}; \quad (4)$$

$$\sum_{|I| \leq k} |A_I^0(x_0, \xi)| \neq 0 \quad \text{for every } \xi \in S^{n-1}, \quad (5)$$

where A_I^0 is the principal part of the symbol $\sigma(A_I)(x, \xi)$ of the operator A_I (see, for example, (7)), $S^{n-1} = \{\xi \in R^n : \sum \xi_j^2 = 1\}$.

Remark. It is easy to show that the definition of $R_{(A_0, \dots, A_{2n})}(x)$ is invariant under infinitely differentiable changes of independent variables; moreover, if $R_{(A_0, \dots, A_{2n})}(x) < \infty$ for every point $x \in \Omega$, then

$$R_L(K) = \sup_K R_{(A_0, \dots, A_{2n})}(x) = k(K) < \infty$$

for every compact set $K \subset \Omega$.

Theorem 1. If $R_L(x) < \infty$ for every point $x \in \Omega$, then for every compact $K \subset \Omega$ there exists a constant $C(K)$ such that the inequality

$$\begin{aligned} \sum_{j=1}^n \left(\|L^{0(j)}u\|_{(\varepsilon(K))}^2 + \|L_{(j)}^0u\|_{(\varepsilon(K)-1)}^2 \right) + \|u\|_{(\varepsilon(K)+2^{1-R_L(K)})}^2 &\leq \\ &\leq C(K) \left(\|Pu\|_{(0)}^2 + \|u\|_{(0)}^2 \right), \quad u \in C_0^\infty(K), \end{aligned} \quad (6)$$

holds, where

$$\varepsilon(K) = \min \left(1, 2^{1-R(K)} (2^{R(K)-1} - 1)^{-1} \right).$$

Theorem 2. If $R_L(x) < \infty$ for every point $x \in \Omega$, then for every compact $K \subset \Omega$, every $s \in \mathbb{R}^1$, there exists a constant $C(K, s)$ such that for every function $u \in D'(\Omega)$ such that $Lu \in H_{(s)}^{\text{loc}}$, an estimate of the form

$$\|\varphi u\|_{(s+\varepsilon(K)+2^{1-R_L(K)})}^2 \leq C(K, s) \left(\|\varphi_1 Pu\|_{(s)}^2 + \|\varphi_1 u\|_{(\gamma)}^2 \right), \quad (7)$$

holds, where the functions $\varphi, \varphi_1 \in C_0^\infty(K)$ and $\varphi_1 \equiv 1$ in a neighborhood of $\text{supp } \varphi$, $\gamma = \text{const} < s + \varepsilon(K) + 2^{1-R_L(K)}$.

Theorem 3. If $R_L(x) < \infty$ for every point x of the manifold M , where L is a second-order differential operator with nonnegative characteristic form, defined on M , then L is a hypoelliptic operator.

The proof of Theorem 1, from which Theorems 2 and 3 can be obtained by known methods (see, for example, (10)), is based on the following auxiliary assertions.

Lemma 1 (energy estimate). For every compact $K \subset \Omega$ and every $s \geq 0$ there exists a constant $C(K, s)$ such that for every μ ($0 < \mu < 1$) the inequality

$$\begin{aligned} \sum_{j=1}^n \left(\|L^{0(j)}u\|_{(s)}^2 + \|L_{(j)}^0u\|_{(s-1)}^2 \right) + \|(X_0 + L_{(j)}^{0(j)})u\|_{(s-1/2)}^2 &\leq \\ &\leq C(K, s) \left\{ \frac{1}{\mu} \|Lu\|_{(0)}^2 + \mu \|u\|_{(2s)}^2 + C_\mu \|u\|_{(0)}^2 \right\}, \quad u \in C_0^\infty(K). \end{aligned} \quad (8)$$

For every $s \in \mathbb{R}^1$ in the domain Ω introduce the pseudodifferential operator $\mathcal{E}_{(s)}$ with symbol

$$\sigma(\mathcal{E}_{(s)})(x, \xi) = \varphi(x)(1 + |\xi|^2)^{s/2},$$

where the function $\varphi(x) \in C_0^\infty(\Omega)$ and $\varphi \equiv 1$ in a neighborhood of the compact set K .

Consider the system $(Q_0, \dots, Q_{2n})$ of first-order pseudodifferential operators, where $Q_j = L^{0(j)}$ ($j = 1, \dots, n$); $Q_0 = X_0 + L_{(j)}^{0(j)}$, $Q_{n+j} = L_{(j)}^0 \mathcal{E}_{(-1)}$ ($j = 1, \dots, n$).

Lemma 2. For every compact $K \subset \Omega$, every integer $k \geq 1$, and every s ($0 \leq s < 1/2^{k-1}$) there exists a constant $C(K, k, s)$ such that for

for any μ ($0 < \mu < 1$) the inequality

$$\sum_{|I|=k} \|Q_{Iu}\|_{(s-1+2^{1-k})}^2 \leq C(K, k, s) \times \left\{ \frac{1}{\mu} \|Lu\|_{(0)}^2 + \mu \|u\|_{(2s)}^2 + \mu \|u\|_{(2^{k-1}s)}^2 + C_\mu \|u\|_{(0)}^2 \right\}, \quad u \in C_0^\infty(K). \quad (9)$$

From the condition $R_L(K) < \infty$ it follows that there exists a neighborhood $O_x \subset \Omega$ of the point x and a finite system of operators $(Q_{I_1}, \dots, Q_{I_l})$, ($|I_j| \leq R$ for any $j \leq l$), which is elliptic in the domain U_x . Therefore the following is valid.

Lemma 3. If $R_L(x) < \infty$, then there exists a neighborhood $U_x \subset \Omega$ of the point x such that an estimate of the form

$$\|u\|_{(s+2^{1-R_L(x)})}^2 \leq C_1 \left\{ \sum_{|I| \leq R_L(x)} \|Q_{Iu}\|_{(s-1+2^{1-|I|})}^2 + \|u\|_{(s)}^2 \right\}, \quad u \in C_0^\infty(O_x). \quad (10)$$

From Lemmas 1-3 follows the proof of Theorem 1.

In papers ^(8,9), for a scalar pseudodifferential operator P and any compact set $K \subset \Omega$, algebraic necessary and sufficient conditions were obtained for the existence of estimates of the form:

$$\|u\|_{(m-\delta)}^2 \leq C(K) \{ \|Pu\|_{(0)}^2 + \|u\|_{(0)}^2 \}, \quad u \in C_0^\infty(K), \quad (11)$$

where m is the order of the operator P and $0 \leq \delta < 3/4$.

We shall consider the class of pseudodifferential operators P of order m , satisfying the following conditions:

1. $p^0(x, \xi) \geq 0$ for any point $(x, \xi) \in \Omega \times S^{n-1}$.
2. $p^1(x, \xi) = -\overline{p^1(x, \xi)}$ for any point $(x, \xi) \in N$, where

$$N = \{(x, \xi) \in \Omega \times S^{n-1}; p^0(x, \xi) = 0\},$$

and $p^\nu(x, \xi)$ are the terms homogeneous of order $m - \nu$ in the asymptotic expansion in ξ of the symbol $\sigma(P)(x, \xi)$ of the operator P as $\xi \rightarrow \infty$ (see, for example, (7)).

Applying the method of localization of pseudodifferential operators proposed in paper (8), we obtain algebraic necessary and sufficient conditions for the existence, for any compact set $K \subset \Omega$, in the class of operators satisfying conditions 1, 2, of estimates of the form

$$\|\varphi u\|_{(m+s-1)}^2 \leq C(K, s) \{ \|\varphi_1 P u\|_{(s)}^2 + \|\varphi_1 u\|_{(\gamma)}^2 \} \quad (12)$$

for any function $u \in D'(\Omega)$ such that $P u \in H_{(s)}^{\text{loc}}$, where the functions $\varphi, \varphi_1 \in C_0^\infty(K)$; $\varphi_1 \equiv 1$ in a neighborhood of $\text{supp } \varphi$; $\gamma = \text{const} < m + s - 1$.

For any point (x, η) of the characteristic manifold N , consider the symmetric matrix

$$\mathfrak{A}(x, \eta) = \begin{pmatrix} p^{0(kj)}(x, \eta) & p_{(j)}^{0(k)}(x, \eta) \\ p_{(k)}^{0(j)}(x, \eta) & p_{(kj)}^0(x, \eta) \end{pmatrix} \quad (k, j = 1, \dots, n),$$

where for any multiindices α, β ($|\alpha| + |\beta| = 2$)

$$p_{(\beta)}^{\nu(\alpha)}(x, \xi) = \partial_x^\beta \partial_\xi^\alpha p^\nu(x, \xi).$$

By virtue of condition 1 the matrix $\mathfrak{A}(x, \eta)$ is positive semidefinite at any point $(x, \eta) \in N$. Let

$$\{Y_j = (a_1^j, \dots, a_n^j, b_1^j, \dots, b_n^j), \quad j = 1, \dots, 2n\}$$

be an orthonormal system of eigenvectors of the matrix $\mathfrak{A}(x, \eta)$, and let

$$\{Y_j^+, \quad j \leq s(x, \eta)\}$$

be the set of eigenvectors corresponding to the positive eigenvalues of this matrix;

$$I(x, \eta) \equiv |p_{(j)}^{0(j)}(x, \eta) - i p^1(x, \eta)| + \sum_{j,k}^{S(x, \eta)} \left| \sum_{l=1}^n (a_l^j b_l^k - a_l^k b_l^j) \right|.$$

Theorem 4. A necessary and sufficient condition for the existence, for the operator P with conditions 1, 2, for every compact $K \subset \Omega$, of the estimate

$$\|u\|_{(m-1)}^2 \leq C(K) \{ \|P u\|_{(0)}^2 + \|u\|_{(0)}^2 \}, \quad u \in C_0^\infty(K), \quad (13)$$

is the fulfillment of the inequality

$$I(x, \eta) > 0 \quad \text{for every point } (x, \eta) \in N. \quad (14)$$

Theorem 5. If the pseudodifferential operator P satisfies conditions 1, 2 and (14), then for every compact $K \subset \Omega$ there exists a constant $C(K)$ such that the inequality

$$\begin{aligned} \|u\|_{(m-1)}^2 + \sum_{s=1}^n \|P^{(s)}u\|_{(1/2)}^2 + \|P_{(s)}u\|_{(-1/2)}^2 &\leq \\ &\leq C(K)\{\|Pu\|_{(0)}^2 + \|u\|_{(0)}^2\}, \quad u \in C_0^\infty(K). \end{aligned} \quad (15)$$

From estimate (15) (see, for example, ¹⁰) it follows:

Theorem 6. If the differential operator P with infinitely differentiable coefficients satisfies conditions 1, 2 and (14), then for every compact $K \subset \Omega$ and every $s \in \mathbb{R}^1$ there exists a constant $C(K, s)$ such that for any function $u \in D'(\Omega)$ such that $Pu \in H_{(s)}^{\text{loc}}$, the estimate

$$\|\varphi u\|_{(s+m-1)}^2 \leq C(K, s)\{\|\varphi_1 Pu\|_{(s)}^2 + \|\varphi_1 u\|_{(\gamma)}^2\},$$

holds, where the functions $\varphi, \varphi_1 \in C_0^\infty(K)$ and $\varphi_1 \equiv 1$ on $\text{supp } \varphi$; $\gamma = \text{const} < s + m - 1$.

Theorem 7. A differential operator P satisfying conditions 1, 2 and (14) is hypoelliptic in Ω .

Remark. For a differential operator P of second order of the form (1), condition (14) is equivalent to the following inequality:

$$|X_0(x, \xi)| + \sum_{j,k=1}^N |[X_j, X_k](x, \xi)| > 0 \quad \text{for every point } (x, \xi) \in N.$$

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