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Abstract

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MATHEMATICS

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ON THE CONSTRUCTION OF REGULARIZING ALGORITHMS UNDER RANDOM PERTURBATIONS

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1°. Let X and F be complete Hilbert spaces (real or complex), and let Ω be some probability space ⁽¹⁾, while $x(\omega)$ and $f(\omega)$ are strongly measurable ⁽²⁾ mappings of Ω into X and F , respectively. In the usual way one can define ⁽²⁾ the Banach spaces $L_p(X)$ and $L_p(F)$ of functions summable to the power p with values in X and F . We also consider a uniformly measurable function $A(\omega)$ with values in the space of linear bounded operators from X to F . Let $A(\omega) \in L_{p'}(X \rightarrow F)$, where $1/p + 1/p' = 1$. In this case $A(\omega)$ generates a linear bounded operator from $L_p(X)$ to $L_p(F)$: $y(\omega) = A(\omega)x(\omega)$. We pose the problem of constructing regularizing algorithms (r.a.) ⁽³⁻⁶⁾ for finding preimages under the mapping $A(\omega)$. We shall be interested only in those r.a. which consist of operators from $L_p(F)$ to $L_p(X)$ of a special form—namely, measurable functions with values in the space of bounded mappings from F to X (pointwise r.a.), since usually we deal with sample values of random variables. Note that the problem of finding such r.a. formalizes and generalizes the problem, of great practical interest, of solving the equation

$$Ax = f$$

$f \in F$, $x \in X$, and A a linear mapping from X to F , in the presence of “random” perturbations of f and A .

It is natural to try to obtain r.a. satisfying the required conditions as follows. Let $\{B_\alpha\}$ ($\alpha > 0$) be a family of (linear) operators from F to X , forming an r.a. for the triple (X, A, F) (for the exact definition see ⁽³⁻⁶⁾). Fixing $\omega_0 \in \Omega$, we can construct the family $B_\alpha(\omega_0)$ for the triple $(X, A(\omega_0), F)$. Consequently, one naturally obtains a mapping of Ω into the space of linear mappings from F to X , $B_\alpha(\omega)$.

Under what conditions will $B_\alpha(\omega)$ form an r.a. for $(L_p(X), A(\omega), L_p(F))$? The first result relating to this problem was obtained in ⁽⁶⁾. The corresponding assertion from that work, as applied to our case, may be formulated as follows:

if $A(\omega)$ is a degenerate random variable ($A(\omega) \stackrel{\text{p.v.}}{=} A$) and the “exact” equation relates degenerate random variables, then $B_\alpha \tilde{F}$ converges in measure to X , if $\delta = \|\tilde{F} - F\|_{L_p} \rightarrow 0$, $\alpha \rightarrow 0$, and δ and α are connected by a certain relation. In a later work ⁽⁷⁾ a not quite rigorous attempt was made at a “probabilistic” justification of one special r.a.—the r.a. of A. N. Tikhonov ⁽³⁾. In the present work we shall show that the class of r.a. studied in ^(4,5) (which also includes the r.a. ⁽³⁾) induces (as described above) r.a. for the triple $L_p(X)$, $A(\omega)$, $L_p(F)$. As a consequence, a refinement is obtained of the result formulated in ⁽⁶⁾.

2°. The r.a. studied in ^(4,5) have the form:

$$B_\alpha = \psi(A^*A, \alpha)A^* \quad (\alpha > 0). \quad (1)$$

The operator function $\varphi(A^*A, \alpha)$ is generated by a complex-valued (in general) function $\psi(\lambda, \alpha)$ ($\lambda \in S(A^*A)$) with the following basic properties:

$$\lim_{\alpha \rightarrow 0} \lambda \psi(\lambda, \alpha) = 1, \quad \lambda \neq 0; \quad (2)$$

$$\sup_{\lambda \in S(A^*A)} |\psi(\lambda, \alpha)| \sqrt{\lambda} = K_\alpha < \infty \quad (\alpha > 0). \quad (3)$$

Let us suppose, in addition, that $\psi(\lambda, \alpha)$ is continuous for $\lambda \geq 0$, $\alpha > 0$,

$$\sup_{0 \leq \lambda < \infty} |\psi(\lambda, \alpha)| \sqrt{\lambda} = K_\alpha, \quad \lambda \psi(\lambda, \alpha) \geq \lambda \psi(\lambda, \alpha'), \quad \alpha' \geq \alpha. \quad (4)$$

These restrictions are not especially constraining, since all r.a. used in computations satisfy these conditions ⁽⁵⁾. Formally one may form (for almost all ω) an expression of the form

$$B_\alpha(\omega) = \psi(A^*(\omega)A(\omega), \alpha)A^*(\omega). \quad (5)$$

Theorem 1. The operators (5) form an r.a. for the triple $(L_p(X), A(\omega), L_p(F))$.

Theorem 2. If $\|\tilde{A} - A\|_p \rightarrow 0$, then $\|\tilde{B}_\alpha - B\|_q \rightarrow 0$ for fixed α ($\|\tilde{A} - A\|_p = (\int_\Omega \|\tilde{A}(\omega) - A(\omega)\|^p d\omega)^{1/p}$, $q \geq 1$, and $\|\tilde{B}_\alpha - B\|_q$ is defined correspondingly).

Proofs of the theorems.

A. $B_\alpha(\omega)$ is measurable for any α . Choose a system of sets $\{g_n\} \subseteq \Omega$, $g_n \subset g_{n+1}$, such that $\mu(g_n) \rightarrow 1$, $n \rightarrow \infty$, and $\|A(\omega)\| \leq C_n < \infty$, $\omega \in g_n$. This can be done since $A(\omega) \in L_p$. The union of the spectra of the operators $A^*(\omega)A(\omega)$, $\omega \in g_n$, is evidently located on the segment $[0, C_n^2]$. Since, by (4), $\psi(\lambda, \alpha)$ is continuous in λ on $[0, C_n^2]$, it may be regarded as the limit of a uniformly convergent sequence

of polynomials $p_m(\lambda, \alpha)$. The corresponding functions $P_m(A^*(\omega)A(\omega), \alpha)A^*(\omega)$ are obviously measurable, and for any $\omega \in g_n$, $P_m \rightarrow B_\alpha$ (uniformly). Therefore

$$[B_\alpha(\omega)]_n = \begin{cases} B_\alpha(\omega), & \omega \in g_n, \\ 0, & \omega \notin g_n, \end{cases}$$

is measurable. But $B_\alpha(\omega)$ is the limit in measure of $[B_\alpha(\omega)]_n$, and therefore is also measurable.

B. In view of (4) one may assert that $\text{vrai sup}_\omega \|B_\alpha\| \leq K_\alpha$, and therefore

$$B_\alpha(\omega) \in L_p \quad (p \geq 1).$$

C. Introduce the function $P(\omega)$, where $P(\omega_0)$ is the orthoprojector onto $\ker A^*(\omega_0)A(\omega_0)$.

Consider the element $P(\omega)x(\omega)$, $x(\omega) \in L_p$. Using the fact that $P(\omega)$ is the strong limit of operator polynomials in $A^*(\omega)A(\omega)$, one may show that $P(\omega)x(\omega)$ is measurable and $P(\omega)x(\omega) \in L_p$.

D. The rest of the proof follows the usual scheme ⁽³⁻⁶⁾. Let $x(\omega)$ be some element of the nonempty preimage $F(\omega)$. Then

$$\begin{aligned} & \|B_\alpha(\omega)\tilde{F}(\omega) - P(\omega)x(\omega)\|_p \leq \\ & \leq \|B_\alpha(\omega)A(\omega)x(\omega) - P(\omega)x(\omega)\|_p + K_\alpha\|\tilde{F}(\omega) - F(\omega)\|_p. \end{aligned} \quad (6)$$

Since $\{B_\alpha\}$ is an r.a. for the triple (X, A, F) , for almost all ω

$$\|B_\alpha(\omega)A(\omega)x(\omega) - P(\omega)x(\omega)\|_X \rightarrow 0,$$

and, moreover, by (4) the convergence is monotone. From the theorem on termwise integration of monotone sequen-

it follows from the inequalities that $\|B_\alpha(\omega)A(\omega)z(\omega) - P(\omega)x(\omega)\|_p \rightarrow 0$. Finally, from (6),

$$\lim_{\substack{\alpha \rightarrow 0 \\ K_\alpha\|\tilde{F}-F\|_p \rightarrow 0}} \|B_\alpha(\omega)\tilde{F}(\omega) - P(\omega)x(\omega)\|_p = 0.$$

Theorem 1 is proved.

To prove Theorem 2, we first show that $\tilde{B}_\alpha \rightarrow B_\alpha$ in measure. This is done analogously to item A. From Chebyshev's inequality (1) and item B, the assertion of Theorem 2 follows.

Let us also note that for $p = 2$, $L_2(X)$ and $L_2(F)$ are Hilbert spaces, and the r.a. for our triple can be constructed directly, using scheme (5). It turns out that r.a. constructed in this way coincide with those constructed by formula (5). In this same case one can give a simple probabilistic meaning to the optimal r.a. (5).

Example. Consider the integral equation

$$A\varphi = \int_a^b K(s, \tau)\varphi(\tau) d\tau = \eta(s). \quad (7)$$

Suppose that this equation is solvable in the space $L_2(a, b)$. In the present case $X = F = L_2(a, b)$. Suppose that in fact what is known to us is not $\eta(s)$, but $\tilde{\eta}(s) = \eta(s) + \xi(s)$, where ξ is a realization of some random process, whose sample trajectories belong to $L_2(a, b)$. Suppose it is known that

$$\left(\int_{\Omega} \|\xi(s)\|_{L_2}^p d\omega \right)^{1/p} \leq \delta.$$

Applying (pointwise in ω) any r.a. for $(L_2(a, b), A, L_2(a, b))$ satisfying (2)–(4), for example $\psi(\lambda, \alpha) = 1/(\lambda + \alpha)$, to $\tilde{\eta}(s)$, we may assert that the random variable $\psi(A^*A, \alpha)A^*\tilde{\eta}$ converges in the norm $L_p(L_2(a, b))$ to the exact solution of (7) under the condition $K_\alpha\delta \rightarrow 0$, $\alpha \rightarrow 0$.

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