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Abstract

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MATHEMATICS

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REGULAR POINTS, THE SPECTRUM, AND EIGENFUNCTIONS OF NONLINEAR OPERATORS

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The aim of the present note is to generalize a number of known facts from the spectral theory of linear operators to certain classes of nonlinear operators. In § 1 we study the location of regular points and points of the spectrum of nonlinear operators in Hilbert space; in § 2, the structure of the spectrum and of the set of regular values of completely continuous operators in Banach spaces. In § 3 a theorem is obtained on the existence of a continuum of eigenfunctions for the problem $\lambda F(u) = \Phi(u)$. In § 4 the results obtained are applied to nonlinear elliptic boundary-value problems.

Eigenfunctions and eigenvalues of nonlinear operators have previously been studied in a number of papers. For detailed information on them see ^(1,2).

Let E be a Banach space; $A : E \rightarrow E$ an operator, defined on all of E , linear or nonlinear. We recall the known definitions. A number λ is called a regular value of the operator A if the operator $R_\lambda = (A - \lambda I)^{-1} = A_\lambda^{-1}$ exists, is defined on all of E , and satisfies the Lipschitz condition

$$\|R_\lambda(x) - R_\lambda(y)\| \leq K\|x - y\|, \quad \forall x, y \in E.$$

Here I is the identity operator in E , $K = \text{const}$. From the existence of R_λ it follows at once that $K > 0$. The set of all nonregular values in the complex plane is called the spectrum of the operator A . If $A(\theta) = \theta$, $A(x_\lambda) = \lambda x_\lambda$, $x_\lambda \neq \theta$, then the vector x_λ is called an eigenfunction of the operator A , corresponding to the eigenvalue λ . Obviously, eigenvalues are points of the spectrum. An operator A is called completely continuous if it is continuous and compact. A is called positively homogeneous of degree $\nu > 0$ if for all $x \in E$ and all $\mu \geq 0$ the equality

$$A(\mu x) = \mu^\nu A(x)$$

holds. For $\nu = 1$ the operator A is called positively homogeneous. If $A(\mu x) = \mu A(x)$, $\forall x \in E$ and all real μ , then A is called homogeneous. Let E^* be the space conjugate to E , and let $\Phi : E \rightarrow E^*$ be an operator defined on all of E . If

there exists a functional $\varphi(x)$, defined on all of E , whose Fréchet derivative at every point $x \in E$ is equal to $\Phi(x)$, then Φ is called a potential operator, and φ a potential of the operator Φ . The operator $\Phi : E \rightarrow E^*$ is called strictly monotone in E if

$$(\Phi(x) - \Phi(y), x - y) > 0, \quad \forall x, y \in E, x \neq y.$$

Here (z, x) is the value of the functional $z \in E^*$ on the element $x \in E$.

1. Let H be a Hilbert space, and let $A : H \rightarrow H$ be an operator defined on all of H . Consider the numbers

$$m = \inf_{\substack{x, y \in H \\ x \neq y}} \frac{(A(x) - A(y), x - y)}{\|x - y\|^2}; \quad M = \sup_{\substack{x, y \in H \\ x \neq y}} \frac{(A(x) - A(y), x - y)}{\|x - y\|^2}. \quad (1)$$

Theorem 1. Let A be a continuous operator in H , and suppose the form $(A(x) - A(y), x - y)$ takes real values for all $x, y \in H$.

- a) Every complex number $\lambda = \alpha + \beta i$, $\beta \neq 0$, is a regular value of the operator A ; b) the spectrum of the operator A lies on the segment $[m, M]$;
- c) if the numbers m, M in formulas (1) are finite and the operator A satisfies the Lipschitz condition

$$\|A(x) - A(y)\| \leq \max\{|m|, |M|\} \|x - y\|, \quad \forall x, y \in H,$$

then at least one of the numbers m, M is a point of the spectrum of the operator A .

Theorem 2. Let A be a completely continuous positively homogeneous potential operator of degree $\nu > 0$ in H , and let the form $(A(x), x)$ take real values for all $x \in H$.

- a) If $\nu \neq 1$ and there exists an element $u_1 \in H$ such that $(A(u_1), u_1) > 0$, then every positive number is an eigenvalue of the operator A ; b) if $\nu \neq 1$ and there exists an element $u_2 \in H$ such that $(A(u_2), u_2) < 0$, then every negative number is an eigenvalue of the operator A ; c) if A is an even operator (i.e. $A(-u) = A(u)$, $\forall u \in H$, $A(x) \neq \theta$ in H and $\nu \neq 1$, then every real number $\gamma \neq 0$ is an eigenvalue of the operator A ; d) if $\nu = 1$, then the numbers

$$m_0 = \inf_{\|x\|=1} (A(x), x),$$

$$M_0 = \sup_{\|x\|=1} (A(x), x)$$

are finite and at least one of them is an eigenvalue of the operator A .

The following results are also valid in Banach spaces.

2. Let E be a real Banach space, $A : E \rightarrow E$ an operator defined on all of E .

Lemma. Let A be a completely continuous homogeneous operator in E . In order that the number $\lambda \neq 0$ be a regular value of the operator A , it is necessary and sufficient that there exist a number $c > 0$ such that

$$\|A_\lambda(x) - A_\lambda(y)\| \geq c\|x - y\|, \quad \forall x, y \in E, \quad A_\lambda(x) = A(x) - \lambda x.$$

With the aid of the lemma one proves

Theorem 3. If zero is a point of the spectrum of a completely continuous homogeneous operator A in E , then the set of regular values of the operator A is an open set, and the spectrum is a closed set.

3. In this section E is a real reflexive Banach space, E^* its conjugate space, and $F : E \rightarrow E^*$, $\Phi : E \rightarrow E^*$ are operators defined on all of E . Consider the question of the existence of nonzero solutions (eigenfunctions) of the equation $\lambda F(x) = \Phi(x)$ with parameter λ .

Theorem 4 (cf. ⁽¹⁾, p. 343; ⁽²⁾). Let F be a strictly monotone potential operator with potential f , $F(\theta) = \theta$,

$$\lim_{\|x\| \rightarrow \infty} f(x) = +\infty.$$

Let Φ be a completely continuous potential operator, $\Phi(\theta) = \theta$. Then in the space E there exists a continuum of distinct nonzero solutions of the equation

$$\lambda F(x) = \Phi(x)$$

(these solutions may correspond to one or to different values of the parameter λ).

If, in addition to the conditions of the theorem, the operators F and Φ are positively homogeneous of degree $\gamma > 0$, then the numbers

$$\begin{aligned} \tilde{m}_0 &= \inf_{x \in X} (\Phi(x), x), \quad \tilde{M}_0 = \\ &= \sup_{x \in X} (\Phi(x), x) \end{aligned}$$

are finite, and for λ equal to at least one of them the problem

$$\lambda F(x) = \Phi(x)$$

has a nonzero solution. Here

$$X = \{x : (F(x), x) = 1, x \in E\}.$$

4. We shall dwell on some applications of the results obtained above. We restrict ourselves here only to the application of Theorems 2 and 4 to the study of the existence of eigenfunctions and eigenvalues in nonlinear elliptic boundary-value problems.

Let Ω be a bounded domain of the n -dimensional space R^n with piecewise smooth boundary $\partial\Omega$, $1 < p < \infty$. On the Sobolev space $\dot{W}_p^m(\Omega)$ consider the functionals

$$f(u) = \int_{\Omega} \psi(x, u, \dots, D^m u) dx, \quad \varphi(u) = \int_{\Omega} \omega(x, u, \dots, D^{m-1} u) dx. \quad (2)$$

We shall say that the function $\delta(x, u, \dots, D^\alpha u, \dots, D^k u)$ satisfies the Carathéodory conditions if, for almost all $x \in \Omega$, it is continuous jointly in the variables $D^\alpha u$ ($|\alpha| \leq k$) and is measurable in x in Ω for any

values $D^\alpha u$. Denote the partial derivatives of the function ψ with respect to $D^\alpha u$ by ψ_α , and those of the function ω with respect to $D^\sigma u$ by ω_σ .

Condition I. There exist $\psi_\alpha, \omega_\sigma$ for all α, σ , $|\alpha| \leq m$, $|\sigma| \leq m-1$, and the functions $\psi, \omega, \psi_\alpha, \omega_\sigma$ satisfy the Carathéodory conditions and the inequalities

$$\begin{aligned} |\psi(x, u, \dots, D^m u)| &\leq c \left[k_1(x) + \sum_{|\gamma| \leq m} |D^\gamma u|^{p_\gamma} \right]; \\ |\omega(x, u, \dots, D^{m-1} u)| &\leq c \left[k_1(x) + \sum_{|\beta| \leq m-1} |D^\beta u|^{q_\beta} \right]; \\ |\psi_\alpha(x, u, \dots, D^m u)| &\leq c \left[k_2(x) + \sum_{|\gamma| \leq m} |D^\gamma u|^{p_{\alpha\gamma}} \right]; \\ |\omega_\sigma(x, u, \dots, D^{m-1} u)| &\leq c \left[k_2(x) + \sum_{|\beta| \leq m-1} |D^\beta u|^{p_{\sigma\beta}} \right], \end{aligned}$$

where $k_1(x) \in L_1(\Omega)$; $k_2(x) \in L_{p'}(\Omega)$; $1/p + 1/p' = 1$; $c = \text{const}$; $0 \leq p_\gamma \leq np/[n - (m - |\gamma|)p]$; $0 \leq p_{\alpha\gamma} \leq [n(p-1) + (m - |\alpha|)p]/[n - (m - |\gamma|)p]$, if $n > (m - |\gamma|)p$; $p_\gamma \geq 0$, $p_{\alpha\gamma} \geq 0$ are arbitrary numbers if $n \leq (m - |\gamma|)p$; $0 \leq q_\beta \leq np/[n - (m - |\beta|)p]$, $0 \leq p_{\sigma\beta} \leq [n(p-1) + (m - |\sigma|)p]/[n - (m - |\beta|)p]$, if $n > (m - |\beta|)p$; $q_\beta \geq 0$, $p_{\sigma\beta} \geq 0$ are arbitrary numbers if $n \leq (m - |\beta|)p$.

Condition II. For any $u, v \in \dot{W}_p^m(\Omega)$, $u \neq v$, the inequality

$$\int_{\Omega} \sum_{|\alpha| \leq m} [\psi_\alpha(x, u, \dots, D^m u) - \psi_\alpha(x, v, \dots, D^m v)] D^\alpha(u - v) dx > 0$$

holds.

In the space $\dot{W}_p^m(\Omega)$ let us consider the question of the existence of nonzero solutions (eigenfunctions) in the nonlinear elliptic boundary-value problem with parameter λ

$$\lambda \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha \psi_\alpha(x, u, \dots, D^m u) = \sum_{|\sigma| \leq m-1} (-1)^{|\sigma|} D^\sigma \omega_\sigma(x, u, \dots, D^{m-1} u)$$

$$D^i u|_{\partial\Omega} = 0, \quad |i| \leq m-1. \quad (3)$$

Theorem 5. Suppose that Conditions I, II hold and

$$\begin{aligned} & \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha \psi_\alpha(x, u, \dots, D^m u)|_{u=0} = \\ & = \sum_{|\sigma| \leq m-1} (-1)^{|\sigma|} D^\sigma \omega_\sigma(x, u, \dots, D^{m-1} u)|_{u=0} = 0 \end{aligned}$$

in the space $W_{p'}^m(\Omega)$. Let the functional $f(u)$, defined by equalities (2), be such that $\lim_{\|u\| \rightarrow \infty} f(u) = +\infty$, $\|u\|_1 = \|u\|_{\dot{W}_p^m(\Omega)}$. Then in the space $\dot{W}_p^m(\Omega)$ there exists a continuum of distinct nonzero solutions of problem (3), and these solutions may correspond to one and the same or to different values of the parameter λ .

If, in addition to the assumptions of the theorem, the functionals f, φ , defined by equalities (2), satisfy the conditions $f(\mu u) = \mu^{\nu+1} f(u)$, $\varphi(\mu u) = \mu^{\nu+1} \varphi(u)$, $\nu > 0$, for all $u \in \dot{W}_p^m(\Omega)$ and all $\mu \geq 0$, then the numbers

$$\begin{aligned} \widetilde{m}_0 &= \inf_{u \in X} \sum_{|\sigma| \leq m-1} \int_{\Omega} \omega_\sigma(x, u, \dots, D^{m-1} u) D^\sigma u \, dx, \\ \widetilde{M}_0 &= \sup_{u \in X} \sum_{|\sigma| \leq m-1} \int_{\Omega} \omega_\sigma(x, u, \dots, D^{m-1} u) D^\sigma u \, dx \end{aligned}$$

are finite, and for λ equal to at least one of them, problem (3) has nonzero solutions. Here

$$X = \left\{ u : u \in \dot{W}_p^m(\Omega), \quad \sum_{|\alpha| \leq m} \int_{\Omega} \psi_\alpha(x, u, \dots, D^m u) D^\alpha u \, dx = 1 \right\}.$$

We now give an application of Theorem 2. Consider the following conditions.

Condition III. The functions $a_{\alpha\delta}(x)$, defined in Ω , are such that the form

$$a(u, v) = \sum_{|\alpha|, |\delta| \leq m} \int_{\Omega} a_{\alpha\delta} D^{\alpha} u D^{\delta} v \, dx$$

is symmetric on $\dot{W}_2^m(\Omega)$ and satisfies the inequalities

$$\mu'_0 \|u\|_1^2 \geq a(u, u) \geq \mu_0 \|u\|_1^2, \quad \forall u \in \dot{W}_2^m(\Omega),$$

where μ'_0, μ_0 are constants, $\mu_0 > 0$, $\|u\|_1 = \|u\|_{\dot{W}_2^m(\Omega)}$.

Condition IV. The functions ω_{σ} exist for all σ , $|\sigma| \leq m-1$, $\omega_{\sigma} = \partial\omega/\partial z_{\sigma}$, $z_{\sigma} = D^{\sigma}u$, and moreover ω, ω_{σ} satisfy the Carathéodory conditions and the inequalities

$$|\omega(x, u, \dots, D^{m-1}u)| \leq c \left[k_1(x) + \sum_{|\beta| \leq m-1} |D^{\beta}u|^{q_{\beta}} \right];$$

$$|\omega_{\sigma}(x, u, \dots, D^{m-1}u)| \leq c \left[k_2(x) + \sum_{|\beta| \leq m-1} |D^{\beta}u|^{p_{\sigma\beta}} \right],$$

where $k_1(x) \in L_1(\Omega)$, $k_2(x) \in L_2(\Omega)$, $0 \leq q_{\beta} < 2n/[n - 2(m - |\beta|)]$, $0 \leq p_{\sigma\beta} \leq [n + 2(m - |\sigma|)]/[n - 2(m - |\beta|)]$, if $|\beta| > m - n/2$; $q_{\beta} \geq 0$, $p_{\sigma\beta} \geq 0$ are arbitrary numbers if $|\beta| \leq m - n/2$.

Condition V. The functional

$$\varphi(u) = \int_{\Omega} \omega(x, u, \dots, D^{m-1}u) \, dx$$

takes only real values and satisfies the condition

$$\varphi(\mu u) = \mu^{\nu+1} \varphi(u), \quad \forall u \in \dot{W}_2^m(\Omega), \quad \forall \mu \geq 0.$$

Here the number ν is fixed, $\nu > 0$, $\nu \neq 1$.

Consider the boundary-value problem with parameter λ

$$\lambda \sum_{|\alpha|, |\delta| \leq m} (-1)^{|\delta|} D^{\delta}(a_{\alpha\delta}(x) D^{\alpha}u) = \sum_{|\sigma| \leq m-1} (-1)^{|\sigma|} D^{\sigma} \omega_{\sigma}(x, u, \dots, D^{m-1}u), \quad (4)$$

$$D^i u|_{\partial\Omega} = 0, \quad |i| \leq m-1.$$

We shall seek nonzero solutions of problem (4) in the space $\dot{W}_2^m(\Omega)$.

Theorem 6. Suppose Conditions III–V are satisfied.

- a) If there exists an element $u_1 \in \dot{W}_2^m(\Omega)$ such that $\varphi(u_1) > 0$, then for every $\lambda > 0$ problem (4) has a nonzero solution; b) if there exists an element $u_2 \in \dot{W}_2^m(\Omega)$ such that $\varphi(u_2) < 0$, then for every $\lambda < 0$ problem (4) has a nonzero solution; c) if $\varphi(-u) = -\varphi(u)$, $\forall u \in \dot{W}_2^m(\Omega)$, and the functional $\varphi(u) \not\equiv 0$ in the space $\dot{W}_2^m(\Omega)$, then for every $\lambda \neq 0$ problem (4) has a nonzero solution.

Let us also note that under the conditions of Theorem 6 we have

$$\varphi(u) = \frac{1}{\nu + 1} \sum_{|\sigma| \leq m-1} \int_{\Omega} \omega_{\sigma}(x, u, \dots, D^{m-1}u) D^{\sigma}u \, dx \quad \text{for all } u \in \dot{W}_2^m(\Omega).$$

The simplest example of problem (4) is the problem, studied in a number of works,

$$\Delta u = u^2, \quad u|_{\partial\Omega} = 0.$$

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