

PARTIAL INVARIANCE

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Abstract

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MATHEMATICS

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PARTIAL INVARIANCE

The concept of a partially invariant solution, introduced earlier in the theory of group properties of differential equations ⁽¹⁾, serves as the basis for constructing broad classes of particular solutions of differential equations, and also finds application outside this theory, for example in the study of generalized motions in Riemannian spaces ⁽²⁾. In this connection there arises the need to reveal the operational, algorithmic content of the concept of partial invariance and to establish the possibility of working with this concept in the infinitesimal language. The present work is devoted to this goal.

1. Initial definitions. Consider an N -dimensional space E^N of points $x = (x^1, \dots, x^N)$, in which there acts a local r -parameter Lie group G of point transformations $x' = T_a x$ ($x'^i = f^i(x, a)$, where $a = (a^1, \dots, a^r)$), with a basis of the Lie algebra consisting of the operators

$$X_\alpha = \xi_\alpha^i(x) \partial / \partial x^i \quad (\alpha = 1, \dots, r). \quad (1)$$

A manifold $\mathfrak{R}$ in E^N is called the set of points satisfying a system of equations of the form

$$\mathfrak{R}: \quad \psi^\sigma(x) = 0 \quad (\sigma = 1, \dots, s \leq N) \quad (2)$$

with continuously differentiable functions $\psi^\sigma(x)$. It is important to bear in mind that all definitions, constructions, and assertions here and below are understood in the local sense. If $M(x)$ is some matrix depending on x and defined in a neighborhood of a point $x_0 \in E^N$, then $M(x)$ is said to have, at the point x_0 , **generic rank** (abbreviated g.r.) if the rank of $M(x)$ is equal to the rank of $M(x_0)$ for all x from some neighborhood of x_0 . The g.r. of $M(x)$ is defined analogously when the point x ranges over some manifold. The manifold (2) is called **regularly given** if g. r. $\|\partial \psi^\sigma / \partial x^i\| = s$ for points $x \in \mathfrak{R}$. It is clear that the dimension of the regularly given manifold (2) is $\dim \mathfrak{R} = N - s$.

A manifold $\mathfrak{R} \subset E^N$ is called an **invariant** manifold of the group G if $x \in \mathfrak{R}$ implies $T_a x \in \mathfrak{R}$ for all $T_a \in G$. Considering all possible invariant manifolds

$\mathfrak{M}(\mathfrak{R})$ of the group G containing the given manifold $\mathfrak{R}$, one can find the least of such invariant manifolds as the intersection of all manifolds $\mathfrak{M}(\mathfrak{R})$.

Definition 1. The **defect of invariance** δ of a given manifold $\mathfrak{R}$ with respect to the group G (or, briefly, the defect of the pair $(\mathfrak{R}, G)$) is the difference between the dimension of the least invariant manifold $\mathfrak{M}$ of the group G containing $\mathfrak{R}$, and the dimension of $\mathfrak{R}$:

$$\delta = \dim \mathfrak{M} - \dim \mathfrak{R}. \quad (3)$$

If $\mathfrak{R}$ has defect of invariance δ with respect to G , then $\mathfrak{R}$ is called a **partially invariant** manifold (of the group G) of defect δ .

2. Main result. Consider a manifold $\mathfrak{R}$, regularly given by equations (2), and a group G with operators (1). Then the following assertion is valid.

Theorem 1. The defect of invariance of the pair $(\mathfrak{R}, G)$ is equal to the general rank of the matrix $\|X_\alpha \psi^\sigma\|$ on the manifold $\mathfrak{R}$:

$$\delta = \text{o. r. } \|X_\alpha \psi^\sigma(x)\|_{\mathfrak{R}}. \quad (4)$$

Proof. To simplify the exposition it is useful to note that neither the number δ nor the right-hand side of (4) depends on the choice of coordinate system in E^N . Therefore one may assume from the outset that the equations of the given manifold have the form

$$\mathfrak{R}: \quad x^\sigma = 0 \quad (\sigma = 1, \dots, s). \quad (5)$$

It is then convenient to write the coordinates of a point in the form $x = (x_1, x_2)$, where $x_1 = (x^1, \dots, x^s)$, $x_2 = (x^{s+1}, \dots, x^N)$. Then the equality (4) to be proved takes the form

$$\delta = \text{o. r. } \|\xi_\alpha^\sigma(0, x_2)\|. \quad (6)$$

Let us now consider another process for constructing the smallest invariant manifold $\mathfrak{M}$ containing the given $\mathfrak{R}$. Let $\mathfrak{N}_a$ be the manifold obtained from $\mathfrak{R}$ by the transformation T_a . Then it is clear that $\mathfrak{M} = \bigcup_a \mathfrak{N}_a$.

Regarding the parameters a as canonical of the first kind, we see that the equations of $\mathfrak{N}_a$ can be written by means of the functions $x'^i = f^i(x, a)$ defining the transformation T_a :

$$\mathfrak{N}_a: \quad f^\sigma(x, a) = 0 \quad (\sigma = 1, \dots, s). \quad (7)$$

The process of constructing the equations of the manifold $\mathfrak{M} = \bigcup_a \mathfrak{N}_a$ is now reduced to eliminating the parameters a from the system of equations (7). To carry out this process, consider Lie's equations

$$\partial f^\sigma / \partial a^\alpha = \xi_\beta^\sigma(f) V_\alpha^\beta(a) \quad (\sigma = 1, \dots, s; \alpha = 1, \dots, r), \quad (8)$$

where $V_\alpha^\beta(a)$ are auxiliary functions of the local Lie group G , and summation over $\beta = 1, \dots, r$ is assumed. Since the matrix $\|V_\alpha^\beta(a)\|$ is nonsingular, putting $\nu = \text{o.r.} \|\xi_\alpha^\sigma(0, x_2)\|$, we shall have

$$\text{o.r.} \|\partial f^\sigma / \partial a^\alpha\|_{\mathfrak{N}_{-a}} = \text{o.r.} \|\partial f^\sigma / \partial a^\alpha\|_{f_1=0} = \text{o.r.} \|\xi_\alpha^\sigma(0, f_2)\| = \nu. \quad (9)$$

Without loss of generality, one may assume that the rank minor in the matrix (9) is formed by the first ν rows ($\alpha = 1, \dots, \nu$) and columns ($\sigma = 1, \dots, \nu$). Introduce notation for the indices $\sigma' = 1, \dots, \nu$; $\alpha' = 1, \dots, \nu$; $\sigma'' = \nu + 1, \dots, s$; $\alpha'' = \nu + 1, \dots, r$. By the implicit function theorem there exist ν functions

$$a^{\alpha'} = A^{\alpha'}(x, \bar{a}) \quad (\alpha' = 1, \dots, \nu), \quad (10)$$

where $\bar{a} = (a^{\nu+1}, \dots, a^r)$, such that identically in $(x, \bar{a})$ one has

$$f^{\sigma'}(x, A(x, \bar{a}), \bar{a}) = 0 \quad (\sigma' = 1, \dots, \nu), \quad (11)$$

and the system consisting of the first ν equations (7) is equivalent to the system of equations (10). Substituting (10) into the functions $f^{\sigma''}(x, a)$, we obtain the functions

$$F^{\sigma''}(x, \bar{a}) = f^{\sigma''}(x, A(x, \bar{a}), \bar{a}) \quad (\sigma'' = \nu + 1, \dots, s). \quad (12)$$

Consider the manifolds

$$\mathfrak{M}_{\bar{a}} : F^{\sigma''}(x, \bar{a}) = 0 \quad (\sigma'' = \nu + 1, \dots, s). \quad (13)$$

It follows directly from the construction that $\mathfrak{M}_0 \supset \mathfrak{M}$. It will further be shown that $\mathfrak{M}_0$ is an invariant manifold of the group G . Hence it follows that $\mathfrak{M}_0 \supset \mathfrak{M}$, and therefore $\dim \mathfrak{M} \leq \dim \mathfrak{M}_0 = N - s + \nu$. A strict inequality here, however, is impossible, since the number of equations of the manifold...

sion of $\mathfrak{M}$ cannot be greater than $s - \nu$. Therefore

$$\dim \mathfrak{M} = N - s + \nu = \dim \mathfrak{N} + \nu,$$

whence equality (6) is obtained in the form $\delta = \nu$.

The invariance of $\mathfrak{M}_0$ is proved as follows. First, by virtue of the regularity of the specification of $\mathfrak{N}_{-a}$ with the aid of equations (7), it is established that all $\mathfrak{M}_{\bar{a}}$ have one and the same dimension, coinciding with the dimension of $\mathfrak{M}_0$. Then it is proved that the inclusion $\mathfrak{M}_0 \subset \mathfrak{M}_{\bar{a}}$ holds. From these two facts there follows the coincidence $\mathfrak{M}_{\bar{a}} = \mathfrak{M}_0$ and, since

$$\bigcup_{\bar{a}} \mathfrak{M}_{\bar{a}} = \bigcup_a (T_a \mathfrak{M}_0),$$

the invariance of $\mathfrak{M}_0$. We give below only the proof of the indicated inclusion, since the remaining assertions are verified quite simply. The idea of this proof is typical for the theory of local Lie groups: it is established that a certain quantity satisfies a system of equations having the zero solution and zero initial conditions, and the uniqueness theorem for the solution is applied. In the present case this quantity is the vector-function $F = (F^{v+1}, \dots, F^s)$ with components (12).

Differentiating the functions (13) with respect to $a^{\alpha''}$ and using equations (8), we obtain

$$\frac{\partial F^{\sigma''}}{\partial a^{\alpha''}} = \frac{\partial f^{\sigma''}}{\partial a^{\alpha'}} \frac{\partial A^{\alpha'}}{\partial a^{\alpha''}} + \frac{\partial f^{\sigma''}}{\partial a^{\alpha''}} = \xi_{\beta}^{\sigma''}(f) V_{\alpha'}^{\beta}(a) \frac{\partial A^{\alpha'}}{\partial a^{\alpha''}} + \xi_{\beta}^{\sigma''}(f) V_{\alpha''}^{\beta}(a), \quad (14)$$

where $f = f(x, A(x, \bar{a}), \bar{a})$ and $a = (A(x, \bar{a}), \bar{a})$. More precisely, by (11), the vector f has the form

$$f = (0, \dots, 0, F^{v+1}, \dots, F^s, f^{s+1}, \dots, f^N).$$

Next, the derivatives $\partial A^{\alpha'}/\partial a^{\alpha''}$ can be found from the equations obtained by differentiating the identities (11):

$$\frac{\partial f^{\sigma'}}{\partial a^{\alpha'}} \frac{\partial A^{\alpha'}}{\partial a^{\alpha''}} + \frac{\partial f^{\sigma'}}{\partial a^{\alpha''}} = 0, \quad (15)$$

since the matrix $\|\partial f^{\sigma'}/\partial a^{\alpha'}\|$ has an inverse, and, by virtue of (9), also when $F = 0$. Therefore the right-hand sides in (14) are regular functions of the variables F in a neighborhood of the point $F = 0$. Further, these right-hand sides must vanish when $F = 0$, since otherwise we would obtain a contradiction with equality (9). This follows from the fact that the expressions (14) and (15) are obtained as elements of the rows with numbers α'' of the matrix which is formed from $\|\partial f^{\sigma'}/\partial a^{\alpha'}\|$ by adding to the row with number α'' a linear combination of the rows with numbers α' and multipliers $\partial A^{\alpha'}/\partial a^{\alpha''}$. Finally, if $x \in \mathfrak{M}_0$, then, by definition, $F(x, 0) = 0$. Thus the solution of the system of equations (14) can only be the zero solution for $x \in \mathfrak{M}_0$, and this means that for such x also $F(x, \bar{a}) = 0$ for all $\bar{a}$, whence the inclusion $\mathfrak{M}_0 \subset \mathfrak{M}_{\bar{a}}$ follows. The theorem is proved.

Remark 1. As a special case of Theorem 1 we obtain the well-known infinitesimal criterion for the invariance of a manifold. If $\mathfrak{N}$ is invariant, then $\mathfrak{M} \equiv \mathfrak{N}$

and $\delta = 0$, so that (4) becomes

$$X_\alpha \psi^\sigma(x)|_{\mathfrak{N}} = 0 \quad (\alpha = 1, \dots, r; \sigma = 1, \dots, s);$$

conversely, if these equalities hold, then $\delta = 0$, $\dim \mathfrak{M} = \dim \mathfrak{N}$, and from the inclusion $\mathfrak{N} \subset \mathfrak{M}$ there follows $\mathfrak{N} = \mathfrak{M}$, i.e. the invariance of $\mathfrak{N}$.

Remark 2. In applications, the assertion of Theorem 1 is convenient to use as a necessary and sufficient condition for a certain manifold $\mathfrak{N}$ to have the given defect of invariance δ with respect to the group G .

3. Rank of a manifold. Along with the defect δ , the pair $(\mathfrak{N}, G)$ is supplied with another numerical characteristic—the rank of $\mathfrak{N}$ with respect to G . However, the rank is not defined for every $\mathfrak{N}$. Let us introduce the number

$$R = \text{o. p. } \|\xi_\alpha^i(x)\|.$$

Definition 2. The manifold $\mathfrak{N}$ is called **nonsingular** with respect to the group G if

$$\text{o. p. } \|\xi_\alpha^i(x)\|_{\mathfrak{N}} = R.$$

If, however,

$$\text{o. p. } \|\xi_\alpha^i(x)\|_{\mathfrak{N}} < R,$$

then $\mathfrak{N}$ is called **singular** with respect to G .

It is clear that if $\mathfrak{N}$ is a nonsingular manifold, then the smallest invariant manifold $\mathfrak{M}$ of the group G containing $\mathfrak{N}$ will also be nonsingular.

But every nonsingular invariant $\mathfrak{M}$ can be specified by a system of equations of the form $J^\tau(x) = 0$ ($\tau = 1, \dots, \mu$), whose left-hand sides are invariants of the group G , and therefore it has a definite dimension in the space of invariants of this group.

Definition 3. The dimension in the space of invariants of the group G of the smallest invariant manifold $\mathfrak{M}$ containing a nonsingular G -relative manifold $\mathfrak{N}$ is called the **rank** of the manifold $\mathfrak{N}$ relative to G (or, briefly, the rank of the pair $(\mathfrak{N}, G)$). The rank is denoted by ρ .

The rank of a manifold relates its defect of invariance to the dimension of the space of invariants t , equal to the number of functionally independent invariants of the group G . For every nonsingular manifold $\mathfrak{N}$ specified by equations of the form (2), the relation

$$t - \rho = s - \delta, \tag{16}$$

holds, obtained by a simple count of the number of equations defining $\mathfrak{M}$.

4. The reduction problem. Let a manifold $\mathfrak{N}$ and a group G be given, and let the pair $(\mathfrak{N}, G)$ have rank ρ and defect δ . Consider some subgroup $G' \subset G$.

In passing to the subgroup the numbers t, ρ, δ change, i.e. the pair $(\mathfrak{N}, G')$ has another rank ρ' and defect δ' . The law governing the change of these numbers is regulated by the following proposition.

Theorem 2. *In passing to a subgroup, the rank of a given nonsingular manifold does not decrease, and its defect of invariance does not increase, i.e.*

$$\rho' \geq \rho, \quad \delta' \leq \delta.$$

In applications, the problem of classifying manifolds satisfying certain additional conditions (for example, solutions of differential equations) is of interest; it consists in distributing them into classes relative to subgroups of the group G . In this connection, manifolds with smaller rank and smaller defect are, as a rule, easier to find. It is therefore desirable to “sift out” those cases in which, upon passing to a subgroup, the rank of the manifold does not change while its defect decreases. In these cases one says that the manifold is **reduced** to a smaller defect of invariance, or that the phenomenon of reduction of the pair $(\mathfrak{N}, G)$ takes place.

Thus the following arises.

Reduction problem. Let a pair $(\mathfrak{N}, G)$ be given, having rank ρ and defect δ ; it is required to determine whether there exists a subgroup $G' \subset G$ such that the pair $(\mathfrak{N}, G')$ has the same rank $\rho' = \rho$, but a smaller defect $\delta' < \delta$.

At present rather few general theorems on reduction are known that give conditions for the reducibility of manifolds. One such theorem is given in ⁽¹⁾. However, the known theorems concern only manifolds that are solutions of differential equations.

In conclusion, the author considers it his duty to note that the writing of this paper was preceded by a fruitful discussion with N. Kh. Ibragimov.

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1. L. V. Ovsyannikov, *Group properties of differential equations*, Siberian Branch of the Academy of Sciences of the USSR, 1962.
2. N. Kh. Ibragimov, *DAN*, **178**, No. 1 (1968).

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