

CONTINUITY OF THE METRIC PROJECTION AND SOME GEOMETRIC PROPERTIES OF THE UNIT SPHERE IN A BANACH SPACE

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Abstract

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MATHEMATICS

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CONTINUITY OF THE METRIC PROJECTION AND SOME GEOMETRIC PROPERTIES OF THE UNIT SPHERE IN A BANACH SPACE

(Presented by Academician A. N. Tikhonov, 1 VII 1968)

Let E and F be metric spaces: 2^F is the set of all closed subsets of the space F . A mapping $\varphi : E \rightarrow 2^F$ is called a multivalued mapping from E into F . φ is called upper semicontinuous ⁽³⁾ if the set $\{x \in E : \varphi x \subset G\}$ is open in E for every open subset $G \subset F$. We shall call the mapping φ H -upper semicontinuous (or upper semicontinuous in the Hausdorff sense) if $x_n \rightarrow x$ implies

$$\rho(\varphi x_n, \varphi x) = \sup\{\rho(y, \varphi x) : y \in \varphi x_n\} \rightarrow 0^*.$$

Both of these definitions coincide with the usual definition of continuity if φ is single-valued.

Let M be a set in E . Denote by T_M the multivalued mapping $E \rightarrow 2^M$ which assigns to each point $x \in E$ the set

$$T_M x = \{y \in M : \rho(x, y) = \rho(x, M)\};$$

T_M is called the metric projection of E onto M ⁽²⁾. Everywhere in what follows: X is a B -space over the field of real numbers; $S = \{x \in X : \|x\| = 1\}$, $S^* = \{f \in X^* : \|f\| = 1\}$; $x_n \rightharpoonup x_0$, if the sequence $\{x_n\}$ converges weakly to x_0 ; $f_n \rightharpoonup f_0$, if $f_n(x) \rightarrow f_0(x)$ for every $x \in X$.

Definition 1. We shall say that X satisfies condition (EPS) if from $\{x_n\} \subset S$, $x_0 \in S$, $\{f_n\} \subset S^*$, $f_0 \in S^*$ ($f_0 \neq f_n$, $n = 1, 2, \dots$), $f_n(x_n) = f_0(x_0) = 1$, $x_n \rightharpoonup x_0$, $f_n \rightharpoonup f_0$, and $\rho(x_0, H_0 \cap H_n) \rightarrow 0$, $\rho(x_n, H_0 \cap H_n) \rightarrow 0$, where $H_0 = \{x : f_0(x) = 1\}$, $H_n = \{x : f_n(x) = 1\}$, it follows that $x_n \rightarrow x_0$.

Definition 2. We shall say that X satisfies condition (CR) (compactly rotund) (respectively (WCR) (weakly compactly rotund)) if for every $f \in S^*$ the set $\{x : f(x) = 1\} \cap S$ is either empty or compact (respectively, it is either empty or weakly compact).

Definition 3. We shall say that X satisfies condition (HR) (H -rotund) (respectively (WHR) (weakly H -rotund)) if for every $f \in S^*$ the set $\{x : f(x) = 1\} \cap S$ is either empty or satisfies the following condition: from $f(y_n) = 1$, $\rho(y_n, \{x : f(x) = 1\} \cap S) \rightarrow 0$ it follows that

$$\rho(y_n, \overline{\text{co}}\{y_n\} \cap S) \rightarrow 0^{**}$$

(respectively, it is either empty, or satisfies the following condition: from $f(y_n) = 1$, $\rho(y_n, \{x : f(x) = 1\} \cap S) \rightarrow 0$ and the weak compactness of $\{y_n\}$, it follows that $\rho(y_n, \overline{\text{co}}\{y_n\} \cap S) \rightarrow 0$).

If (A) and (B) are some properties of the space X , then by $(A) \wedge (B)$ we shall denote the property of the space X which consists in the fact that X possesses both property (A) and property (B).

Theorem 1. *In order that, in a reflexive B -space X , the metric projection onto every convex closed set be H -semi-*

* By definition, we assume that $\rho(y, \Phi) = \infty$ for every $y \in F$.

** By $\overline{\text{co}}\{y_n\}$ we denote the closed convex hull of the sequence $\{y_n\}$.

continuous from above, it is necessary and sufficient that X satisfy condition $(\text{EPS}) \wedge (\text{HR})$.

A set $M \subset X$ is called Chebyshev ⁽¹⁾ if, for every point $x \in X$, the set $T_M x$ is a singleton.

Corollary 1. In a reflexive B -space X satisfying condition $(\text{EPS}) \wedge (\text{HR})$, the metric projection onto every convex Chebyshev set is continuous.

Theorem 2. In a reflexive B -space X the following assertions are equivalent:

- (a) the metric projection onto every convex closed set is upper semicontinuous;
- (b) X satisfies condition $(\text{EPS}) \wedge (\text{CR})$;
- (c) X satisfies the following condition:

(N) From $\{x_n\} \subset S$, $x_0 \in S$, $\{f_n\} \subset S^*$, $f_0 \in S^*$, $f_n(x_n) = f_0(x_0) = 1$, $x_n \rightarrow x_0$, $f_n \rightarrow f_0$ and $\rho(x_0, H_0 \cap H_n) \rightarrow 0$, $\rho(x_n, H_0 \cap H_n) \rightarrow 0$, where $H_0 = \{x : f_0(x) = 1\}$, $H_n = \{x : f_n(x) = 1\}$, it follows that $x_n \rightarrow x_0$.

Corollary 2. In order that, in a strictly convex and reflexive B -space X , the metric projection onto every convex closed set be continuous, it is necessary and sufficient that X satisfy the following condition:

(N_R) From $\{x_n\} \subset S$, $x_0 \in S$, $\{f_n\} \subset S^*$, $f_0 \in S^*$, $f_n(x_n) = f_0(x_0) = 1$, $f_n \rightarrow f_0$, $\rho(x_0, H_0 \cap H_n) \rightarrow 0$, $\rho(x_n, H_0 \cap H_n) \rightarrow 0$, where $H_0(x : f_0(x) = 1)$, $H_n = \{x : f_n(x) = 1\}$, it follows that $x_n \rightarrow x_0$.

A set $M \subset X$ is called boundedly weakly bicomact if its intersection with every closed ball is weakly bicomact.

Theorem 3. Let X be a separable B -space. Then, in order that in X the metric projection onto every boundedly weakly bicomact convex set be H -upper semicontinuous, it is necessary and sufficient that X satisfy condition $(\text{EPS}) \wedge (\text{WHR})$.

Corollary 3. In a separable H -space X satisfying condition $(\text{EPS}) \wedge (\text{WHR})$, the metric projection onto every boundedly weakly bicomact convex Chebyshev set is continuous.

Theorem 4. In a separable B -space X satisfying condition (WCR) , the following assertions are equivalent:

- (a) the metric projection onto every boundedly weakly bicomact convex set is upper semicontinuous;
- (b) X satisfies condition $(\text{EPS}) \wedge (\text{CR})$;
- (c) X satisfies condition (N) .

According to I. Singer ⁽⁴⁾, a B -space X has the Efimov-Stechkin property if it is reflexive and from $\{x_n\} \subset S$, $x_0 \in S$, $x_n \rightarrow x_0$ it always follows that $x_n \rightarrow x_0$. I. Singer ⁽⁴⁾ showed that in a B -space with the Efimov-Stechkin property the metric projection onto every convex closed set is upper semicontinuous (and hence H -upper semicontinuous); in particular, the metric projection onto every convex Chebyshev set is continuous.

Example. Consider in the space l_2 of real square-summable numerical sequences $x = \{\xi_i\}$ the following norm, equivalent to the original one:

$$\|x\|_{S'} = \inf_{x \in \lambda S'} |\lambda|,$$

where

$$S' = \left\{ x = \{\xi_i\} \in l_2 : \sum_{i=1}^{\infty} \xi_i^2 \leq 1, |\xi_i| \leq \frac{1}{2} \right\}.$$

Denote by $\tilde{l}_2$ the space l_2 with norm $\|x\|_{S'}$. Then $\tilde{l}_2$ is a separable reflexive B -space not possessing the Efimov-Stechkin property (see, for example, ⁽⁴⁾). Moreover, in the space $\tilde{l}_2$ the metric projection onto the closed hyperplane $H = \{x = \{\xi_i\} : \xi_1 = \frac{1}{2}\}$ is not upper semicontinuous

from above. However, it can be shown that $\tilde{l}_2$ satisfies condition $(\text{EPS}) \wedge (\text{HR})$ and, consequently, by Theorem 1, in it the metric projection onto every convex closed set is H -upper semicontinuous; in particular, the metric projection onto every convex Chebyshev set is continuous.

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Note: Figure translations are in progress. See original paper for figures.

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