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Abstract

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MATHEMATICS

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ON THE CARDINALITY OF BICOMPACTA WITH THE FIRST AXIOM OF COUNTABIL- ITY

(Presented by Academician P. S. Aleksandrov on 23 V 1969)

In this paper we solve a problem posed almost half a century ago (in 1923) in the first publication of P. S. Aleksandrov and P. S. Uryson devoted to bicom-
pact topological spaces (see ⁽¹⁻³⁾) (this problem belongs to Pavel Sergeevich
Aleksandrov, as I know from him personally).

Theorem 1. *If a noncountable bicomcompactum satisfies the first axiom of count-
ability, then its cardinality is equal to the cardinality of the continuum.*

Obviously, it is enough to prove that the cardinality of our bicomcompactum does
not exceed the cardinality of the continuum.

This result will turn out to be a consequence of a general theorem. We shall
prepare the formulation of the latter by a number of preliminary assertions.

Let $A \subset X$, where X is a topological space. By $[A]$ is denoted the closure of A
in X , and by $|A|$ the cardinality of the set A . Let $x \in X$. By $\chi(x, X)$ is denoted
the character of the point x in X , and moreover

$$\chi(X) = \sup\{\chi(x, X) : x \in X\}.$$

By τ^+ , where τ is a cardinal number, is denoted the first cardinal number greater
than τ . By $\alpha + 1$ is denoted the transfinite ordinal immediately following the
transfinite ordinal α . A space X is called sequential if the closed sets in X are
exactly those sets which contain the limit of every convergent sequence of their
elements.

Let T be some set and α some transfinite number. A tuple k of length α with
values in T is any mapping of the set of all transfinite ordinals less than α into
 T (see ⁽²⁾). The length of the tuple k is denoted by $l(k)$. Let k be a tuple of
length α and let t be some element of the set T . Then (k, t) is the tuple which

coincides, as a mapping, on all $\alpha' < \alpha$ with k and takes the value t at α . We say that the tuple k is an extension of the tuple k' if the mapping k' is a restriction of the mapping k , and in this case we write $k' \leq k$ (and $k' < k$, if, in addition, $k' \neq k$).

Lemma 1. *If X is a Hausdorff sequential space and $A \subset X$, $|A| \leq 2^\tau$, where $\tau \geq \aleph_0$, then also $|[A]| \leq 2^\tau$.*

Lemma 2. *If X is a sequential space and $A \subset X$, $x \in X$, $x \in [A]$, then there exists a set $A' \subset A$ for which $x \in [A']$ and $|A'| \leq \aleph_0$.*

Lemmas 1 and 2 easily follow from Theorem 3.15 of the article ⁽⁴⁾—the second by induction.

Recall that a topological space X is called (τ, ∞) -compact if from every open covering of the space X one can select a covering of cardinality less than τ (see ^(2,3)). Recall also that the body of a covering is the union of all its elements.

Lemma 3. *Let X be a (τ^+, ∞) -compact T_1 -space, where τ is a cardinal number, $\tau \geq \aleph_0$; let A be a closed subset of X ; let $\chi(x, X) \leq 2^\tau$ for all $x \in A$, and let $|A| \leq 2^\tau$. Then $X \setminus A = \bigcup \{F_t : t \in T\}$, where $|T| \leq 2^\tau$ and each F_t is closed in X .*

For the proof of Lemma 3 it is enough to note that A has in X an external base $\mathfrak{B}$ of cardinality $\leq 2^\tau$ (see ⁽⁵⁾); therefore, by virtue of the closedness ...

A in X and (τ^+, ∞) -compactness of X , the bodies of covers of the set A of cardinality $\leq \tau$, composed of elements of $\mathfrak{B}$, have intersection equal to the set A (we have also taken into account the equality $(2^\tau)^\tau = 2^\tau$ and the fact that X is a T_1 -space).

Lemma 4. *Let τ be a cardinal number; $\tau \geq \aleph_0$; let X be a Hausdorff topological space and let $\chi(x, X) \leq \tau$ for all $x \in X$. Then, if $A \subset X$ and $|A| \leq 2^\tau$, then also $|[A]| \leq 2^\tau$.*

Proof. To each point $x \in [A]$ assign some set $A_x \subset A$ for which $|A_x| \leq \tau$ and $x \in [A_x]$. Let γ_x denote a base of cardinality $\leq \tau$ of the space X at the point x . Put

$$\lambda(x) = \{U \cap A_x : U \in \gamma_x\}, \quad x \in [A].$$

Then $|\lambda(x)| \leq \tau$ and $|U \cap A_x| \leq \tau$ for every $U \in \gamma_x$ and every $x \in [A]$, with $U \cap A_x \subset A$. Hence

$$|\{\lambda(x) : x \in [A]\}| \leq ((2^\tau)^\tau)^\tau = 2^\tau.$$

But $\lambda(x_1) \neq \lambda(x_2)$ whenever $x_1 \neq x_2$, since X satisfies the Hausdorff separation axiom. Thus $|[A]| \leq 2^\tau$.

Lemma 5. *If X is a (τ^+, ∞) -compact space, where $\tau \geq \aleph_0$, and τ' is a singular cardinal number greater than τ (in particular, $\tau' = \tau^+$ is suitable), then for a set $A \subset X$ of cardinality τ' in X there is a point of complete accumulation.*

Lemma 5 is an obvious and well-known assertion.

General theorem 2. *Let cardinal numbers τ , m and a topological space X be given, for which the following is known:*

- (1) $\tau \geq \aleph_0$ and $m = 2^\tau$.
- (2) X is a Hausdorff space.
- (3) If $A \subset X$ and $|A| \leq m$, then also $|[A]| \leq m$.
- (4) If A is closed in X and $|A| \leq m$, then

$$X \setminus A = \bigcup \{d_t(A) : t \in T\},$$

where $d_t(A)$ is closed in X for all $t \in T$, and T is some fixed set of cardinality m , the nature of whose elements is immaterial.

- (5) If $A \subset X$ and $x \in X$, $x \in [A]$, then there exists $A' \subset A$ for which $|A'| \leq \tau$ and $x \in [A']$.
- (6) If $A \subset X$ and $|A| \geq \tau^+$, then there exists a point $x \in X$ such that for every neighborhood O_x of it

$$|O_x \cap A| \geq \tau^+.$$

Then $|X| \leq m$.

In the proof of theorem 2, alongside the fundamental notion of a branching system, introduced in the first works of P. S. Aleksandrov, a very essential role is played by the new notion of a free sequence, which first appeared in (6).

Definition 1. A well-ordered set A , $<$ of points of a topological space X is called a **free sequence** if for every $y \in A$:

$$(j_1) : |\{x \in A : x < y\}| < |A|$$

and

$$(j_2) : [\{x \in A : x < y\}] \cap [\{x \in A : x \geq y\}] = \Lambda.$$

Proof of theorem 2. By $\mathfrak{K}$ denote the set of all tuples with values in T of length less than τ^+ . By $\mathfrak{W}$ denote the set of all singular transfinite numbers less than τ^+ . Obviously, $|\mathfrak{W}| = \tau^+$. The unique empty tuple belongs to $\mathfrak{K}$ —we shall denote it by k_0 . Put

$$\mathfrak{K}_\alpha = \{k \in \mathfrak{K} : l(k) = \alpha\}, \quad \alpha < \tau^+.$$

Then $\mathfrak{K}_\alpha = T^{\{\beta < \alpha\}}$. Since $|\{\beta < \alpha\}| \leq \tau$, $|\mathfrak{K}_\alpha| \leq |T|^\tau = (2^\tau)^\tau = 2^\tau$. Therefore

$$|\mathfrak{K}| = \left| \bigcup \{\mathfrak{K}_\alpha : \alpha < \tau^+\} \right| = \tau^+ \cdot 2^\tau = 2^\tau.$$

By ξ denote some choice function defined on the set of all nonempty subsets of the set X : to $A \subset X$, $A \neq \Lambda$, there corresponds some point $\xi(A) \in A$.

In view of (2) one can define two such mappings φ_1 and φ_2 of the set $\mathfrak{P}$ of all closed subsets of the set X that for every $P \in \mathfrak{P}$: a) $\varphi_1 P \in \mathfrak{P}$ and $\varphi_2 P \in \mathfrak{P}$, b) $\varphi_1 P \cup \varphi_2 P = P$, and c) if $\varphi_1 P = P$ or $\varphi_2 P = P$, then either $|P| = 1$, or $P = \Lambda$.

As a result of the definition by transfinite induction, to every tuple $k \in \mathfrak{K}$ there will be assigned some $F(k) \in \mathfrak{P}$, and the following rules will be satisfied:

I. If $l(k) \in \mathfrak{W}$ and $t \in T$, then $F(k, t)$ is either $\varphi_1 F(k)$ or $\varphi_2 F(k)$, moreover necessarily for some $t' \in T$ and $t'' \in T$.

$$F(k, t') = \varphi_1 F(k), \quad F(k, t'') = \varphi_2 F(k).$$

II. If $\alpha = l(k) < \tau^+$ and $\alpha \notin \mathcal{W}$, then

$$F(k) = \bigcap \{F(k') : k' < k\}$$

and

$$(7) F(k, t) = (d_t[\eta(k)]) \cap F(k), \quad t \in T.$$

Here

$$(8) \eta(k') = \{\xi(F(k'') \setminus F(k'', t)) : (k'', t) \leq k' \text{ and } F(k'') \setminus F(k'', t) \neq \Lambda\}$$

for every $k' \in \mathcal{K}$.

Under this condition (7) is meaningful, for $|\eta(k')| \leq \tau$ and one may apply (3) and (4).

In order that items I and II acquire force and, in accordance with the principle of transfinite induction, define the sets $F(k) \in \mathcal{P}$ for all $k \in \mathcal{K}$, it remains to specify $F(k_0^*)$. We put $F(k_0^*) = X$. (We note that a function F satisfying I and II, whose existence is ensured by our argument, need not be unique—item I leaves us some freedom. We choose some such function and it is precisely this function that we shall denote below by F .)

Put

$$A = \{\xi(F(k) \setminus F(k, t)) : k \in \mathcal{K}, t \in T\}$$

and $B = [A]$. Then $|A| \leq 2^\tau = m$ (since $2^\tau \cdot 2^\tau = 2^\tau$ for $\tau \geq \aleph_0$), and, by (3), $|B| \leq m$.

From I, II, and (4) it follows that for every $k \in \mathcal{K}$

$$(9) F(k) \setminus B \subset \bigcup \{F(k, t) : t \in T\}.$$

It is clear from the definition of F that if $k' < k''$ and $l(k') \in \mathcal{W}$, then

$$(10) F(k'') \subset F(k'), \quad \text{and} \quad F(k'') \neq F(k')$$

provided, moreover, $|F(k')| > 1$.

Important remarks: for any tuple $k \in \mathcal{K}$

$$(11) [\eta(k)] \cap F(k, t) = \Lambda.$$

If $k' < k''$, $k', k'' \in \mathcal{K}$, then

$$(12) \eta(k'') \setminus \eta(k') \subset F(k').$$

Put

$$C = \bigcup \{F(k) : |F(k)| \leq m, k \in \mathcal{K}\}.$$

From $|\mathcal{K}| \leq m$ it follows that $|C| \leq m$. Therefore also $|B \cup C| \leq m$.

Suppose now that $X \setminus (B \cup C) \neq \Lambda$ and

$$(13) x^* \in X \setminus (B \cup C).$$

For every $\alpha < \tau^+$ we shall now define a tuple k_α^* of length α so that the following conditions will be fulfilled: (i_1) If $\alpha' < \beta' < \tau^+$, then $k_{\alpha'}^* < k_{\beta'}^*$; (i_2) $x^* \in F(k_\alpha^*)$; (i_3) $|F(k_\alpha^*)| > 1$ for all $\alpha < \tau^+$. (We note that for the empty tuple k_0^* , which has already been put in correspondence with the transfinite 0, $x^* \in X = F(k_0^*)$.)

Suppose that we have already defined the tuples k_α^* for all $\alpha < \alpha_0$, where $\alpha_0 < \tau^+$, and in such a way that for all transfinite numbers α, β less than α_0 , conditions (i_1) , (i_2) , and (i_3) are satisfied.

If α_0 is a limit transfinite number, then as $k_{\alpha_0}^*$ one must take the tuple of length α_0 whose restrictions are all the tuples k_α^* , where $\alpha < \alpha_0$. Then, by item I,

$$F(k_{\alpha_0}^*) = \bigcap \{F(k_\alpha^*) : \alpha < \alpha_0\} \ni x^*.$$

Obviously, $|F(k_{\alpha_0}^*)| > m > 1$ —otherwise $x^* \in F(k_{\alpha_0}^*) \subset C$, which is not true.

Let $\alpha_0 = \alpha' + 1$. Then, by (9) and (13), we have

$$x^* \in F(k_{\alpha'}^*) \setminus B \subset \bigcup \{F(k_{\alpha'}^*, t) : t \in T\}.$$

Choose $t_{\alpha'} \in T$ so that $x^* \in F(k_{\alpha'}^*, t_{\alpha'})$, and put

$$k_{\alpha_0}^* = (k_{\alpha'}^*, t_{\alpha'}).$$

Obviously, now conditions (i_1) , (i_2) , and (i_3) are satisfied for the collection $\{k_\alpha^* : \alpha \leq \alpha_0\}$ of tuples. The principle of transfinite induction permits us to conclude that there exists a family of tuples $\{k_\alpha^* : \alpha < \tau^+\}$ satisfying conditions (i_1) , (i_2) , and (i_3) . From (i_1) , (i_3) , and (10) it follows that

$$F(k_\alpha^*) \setminus F(k_{\alpha+1}^*) \neq \Lambda,$$

for every $\alpha \in \mathcal{W}$.

Put

$$x_\alpha = \xi(F(k_\alpha^*) \setminus F(k_{\alpha+1}^*))$$

for $\alpha \in \mathcal{W}$ and

$$\mathcal{T} = \{x_\alpha : \alpha \in \mathcal{W}\}.$$

Since $x_{\alpha'} \neq x_{\alpha''}$ if $\alpha' \neq \alpha''$, $\alpha', \alpha'' \in \mathcal{W}$, and $|\mathcal{W}| = \tau^+$, it follows that $|\mathcal{T}| = \tau^+$.

The set $\mathcal{T}$ is naturally well-ordered: put $x_{\alpha'} < x_{\alpha''}$ if and only if $\alpha' < \alpha''$. From (11) and (12), taking into account (8) and the definition of the set $\mathcal{T}$, we conclude that $\mathcal{T}, <$ is a free sequence in X . From (6) it follows that some point $y \in X$ is a point of complete accumulation for $\mathcal{T}$. But then $y \in |\mathcal{T}|$ and, by (5), there exists $\mathcal{T}' \subset \mathcal{T}$ such that $y \in |\mathcal{T}'|$ and $|\mathcal{T}'| \leq \tau$. From $|\{\alpha \in \mathcal{W} : x_\alpha \in \mathcal{T}'\}| \leq \tau$ it follows that there exists $\alpha^* \in \mathcal{W}$ such that, if $x_\alpha \in \mathcal{T}'$, then $\alpha > \alpha^*$, and hence $x_\alpha > x_{\alpha^*}$. Then $[\{x_\alpha \in \mathcal{T} : x_\alpha < x_{\alpha^*}\}] \ni y$ and, since $\mathcal{T}$ is a free sequence,

$$[\{x_\alpha \in \mathcal{T} : x_\alpha < x_{\alpha^*}\}] \cap [\{x_\alpha \in \mathcal{T} : x_{\alpha^*} \leq x_\alpha\}] = \Lambda,$$

whence it follows that $U = X \setminus [\{x_\alpha \in \mathcal{T} : x_{\alpha^*} \leq x_\alpha\}]$ is an open set containing the point y , and moreover $U \cap \mathcal{T} \subset \{x_\alpha \in \mathcal{T} : \alpha < \alpha^*\}$. Hence, $|U \cap \mathcal{T}| \leq |\{x_\alpha \in \mathcal{T} : \alpha < \alpha^*\}| = |\{\alpha : \alpha < \alpha^*\}| \leq \tau$. We have arrived at a contradiction to the fact that y is a point of complete accumulation for $\mathcal{T}$. Thus $X = B \cup C$, and, consequently, $|X| = |B \cup C| \leq m$. Theorem 2 is proved.

Definition 2. We shall call the **compactness index** $\text{ic}(X)$ of a topological space X the least of all such cardinal numbers τ that from every open cover of the space X one can extract a cover of cardinality $\leq \tau$. Obviously,

$$\text{ic}(X) \leq \min(|X|, w X).$$

Definition 3. The **volume** $\nu(X)$ of a topological space X is $\text{ic}(X) \cdot \chi(X)$.

Theorem 3. For every Hausdorff topological space X ,

$$|X| \leq 2^{\nu(X)}.$$

Theorem 4. Let τ be a cardinal number, $\tau \geq \aleph_0$, and let X be a sequentially (τ^+, ∞) -compact Hausdorff space. Then, if $\chi(X) \leq 2^\tau$, then

$$|X| \leq 2^\tau.$$

Corollary 1. For every finally compact Hausdorff space X ,

$$|X| \leq \max(\aleph_0, 2^{\chi(X)}).$$

Corollary 2. If $\chi(X) \leq 2^{\aleph_0}$, where X is a sequential finally compact Hausdorff space, then

$$|X| \leq 2^{\aleph_0}.$$

Definition 4. The **tightness** $\tau(x, X)$ of a topological space X at a point $x \in X$ is the least of the cardinal numbers τ such that, whatever $A \subset X$ may be, if $[A] \ni x$, then there exists a family $\lambda(x, A)$ of subsets of the set A for which $|\lambda(x, A)| \leq \tau$, $|Q| \leq \tau$ for every $Q \in \lambda(x, A)$, and

$$\{x\} = \bigcap \{[Q] : Q \in \lambda(x, X)\}.$$

The number

$$\tau(X) = \sup\{\tau(x, X) : x \in X\}$$

will then be called the tightness of X .

Obviously, for a Hausdorff space X one always has $\tau(x, X) \leq \chi(x, X)$.

From Theorem 2 the following general result then follows.

Theorem 5. For every T_1 -space X ,

$$|X| \leq 2^{\text{ic}(X) \cdot \tau(X) \cdot 2^{\log \chi(X)}},$$

where $\log \chi(X)$ is the least of all cardinal numbers τ such that $\chi(X) \leq 2^\tau$.

Note added in proof. I have proved that the cardinality of every sequential bicomact satisfying Suslin's condition does not exceed the cardinality of the continuum. Further, every sequential bicomact has on an everywhere dense set of points the first axiom of countability (G.C.H.).

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