

ON EQUATIONS ARISING IN THE SUMMATION OF PERTURBATION- THEORY SERIES FOR THE SCATTERING MATRIX

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Abstract

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MATHEMATICAL PHYSICS

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ON EQUATIONS ARISING IN THE SUMMATION OF PERTURBATION-THEORY SERIES FOR THE SCATTERING MATRIX

(Presented by Academician N. N. Bogolyubov on 24 II 1969)

1. We consider a model of a scalar real field with interaction Lagrangian:

$$\mathcal{L}_{\text{int}}(x) = g : \varphi^4(x) :, \quad x = (x^0, x^1, x^2, x^3),$$

where $\varphi(x)$ is a free field. In this model the S -matrix is given by the expression

$$S = T \exp \left(ig \int_{-\infty}^{\infty} : \varphi^4(x) : dx \right). \quad (1)$$

Expression (1) admits an expansion in normal products of the field $\varphi(x)$

$$S = F_0 + \sum_{i=1}^N \int \cdots \int F_N(x_1, \dots, x_N) : \varphi(x_1) \cdots \varphi(x_N) : dx_1 \cdots dx_N. \quad (2)$$

The functions F_N are infinite series of contributions from Feynman diagrams with N external lines ($N = 0, 1, 2, \dots$) ^(1,2), and the problem arises of assigning mathematical meaning to these series.

The following scheme is proposed for solving this problem. From the formal expressions (1) and (2), relations between the functions F_N are derived. These relations are considered in the Euclidean domain as a single equation in a certain Hilbert space. By the sum of the series representing F_N we mean the solution of the equations obtained ^(3,4).

2. Two kinds of relations between the functions F_N are possible.

I. Substituting series (2) into the relation of D. A. Kirzhnits ⁽⁵⁾

$$\frac{d}{dg} S = \frac{i}{g} T \int (\mathcal{L}_{\text{int}}(x) S) dx, \quad (3)$$

reducing the right-hand side of (3) to normal form, varying N times with respect to the field $\varphi(x)$, and averaging over the vacuum, we obtain relations between

Fig. 1

Figure 1: Fig. 1

the functions

$$F_N(p_1, \dots, p_N) = \int dx_1 \dots dx_N \exp \left(i \sum_{i=1}^N p_i x_i \right) \sum_{\text{perm}(x_1 \dots x_N)} F_N(x_1, \dots, x_N)$$

in the Euclidean domain:

$$\begin{aligned} \frac{d}{dg} F_N(p_1, \dots, p_N) = \sum_{s=-2}^2 \sum_{i_1 + \dots + i_{2+s} = 1} \binom{4}{2+s} \prod_{l=1}^{2-s} \int \frac{dk_l}{(2\pi)^4 (k_l^2 + m^2)} (2\pi)^4 \delta(p_{i_1} + \dots \\ \dots + p_{i_{2+s}} - k_1 - \dots - k_{2-s}) F_{N-2s}(k_1, \dots, k_{2-s}, p_1, \dots, \check{p}_{i_1}, \dots, \check{p}_{i_{2+s}}, \dots, p_N). \end{aligned} \quad (4)$$

for

$$\sum_{i=1}^N p_i = 0$$

and the initial condition $F_0(0) = 1$, $F_N(0) = 0$.

Equations (4) are represented graphically as follows:

Fig. 1

II. Acting analogously with the relation (1):

$$\frac{\delta}{\delta \varphi(x)} S = iT \left(\frac{\partial \mathcal{L}_{\text{int}}(x)}{\partial \varphi(x)} S \right),$$

we obtain

$$\begin{aligned} F_N(p_1, \dots, p_N) = 4g \frac{1}{N} \left(\sum_{s=-1}^2 \sum_{i_1 \neq \dots \neq i_{2+s} = 1}^N (1+s)^3 \prod_{l=1}^{2-s} \int \frac{dk_l}{(2\pi)^4 (k_l^2 + m^2)} \right. \\ \times (2\pi)^4 \delta(p_{i_1} + \dots + p_{i_{2+s}} - k_1 - \dots - k_{2-s}) \\ \left. \times F_{N-2s}(k_1, \dots, k_{2-s}, p_1, \dots, \check{p}_{i_1}, \dots, \check{p}_{i_{2+s}}, \dots, p_N) \right) \end{aligned} \quad (5)$$

for

$$\sum_{i=1}^N p_i = 0.$$

Equations (5) in graphical form:

Fig. 2

Figure 2: Fig. 2

Fig. 2

In equation (5), F_0 enters only when $N = 4$. Equations for the functions $F'_N = F_N/F_0$ (F'_N do not contain vacuum loops) can be obtained from (5) if one sets $F_0 = 1$ (the prime is omitted below). Equations (4) and (5) were first obtained in works by one of the authors and were studied in detail in the φ^3 model ⁽⁶⁾.

3. Consider the Hilbert space

$$H = \sum_{N=0}^{\infty} H_N$$

of sequences $f = \{f_N\}_{N=1}^{\infty}$ of symmetric functions $f_N(p_1, \dots, p_N)$ with scalar product

$$(f, g) = \sum_{N=0}^{\infty} (f_N, g_N); \quad (f_N, g_N) = \int \bar{f}_N g_N dP_N, \quad dP_N = \prod_{l=1}^N \frac{dp_l}{(2\pi)^4(p_l^2 + m^2)}.$$

Introduce formal operations acting from one H_N into another:

$$\begin{aligned} (a_s f)_N(p_1, \dots, p_N) &= \sum_{i_1 \neq \dots \neq i_{2+s}=1}^N \int dP_{2-s} (2\pi)^4 \delta(p_{i_1} + \dots + p_{i_{2+s}} \\ &\quad - k_1 - \dots - k_{2-s}) f_{N-2s}(k_1, \dots, k_{2-s}, p_1, \dots, \check{p}_{i_1}, \dots, \check{p}_{i_{2+s}}, \dots, p_N), \\ s &= 0, \pm 1, \pm 2. \end{aligned} \tag{6}$$

We pass to new functions F_N . In equations I,

$$F_N \rightarrow \frac{1}{\sqrt{N!}} F_N;$$

in equations II

$$F_N \rightarrow \frac{1}{\sqrt{(N-1)!}} F_N. \tag{7}$$

Let us rewrite equations I and II, using operations (6) and the substitution (7):

$$\begin{aligned} \text{I. } \frac{d}{dg} F_N(g) = & \left\{ \sqrt{(N+1)(N+2)(N+3)(N+4)} (a_{-2}F)_N(g) \right. \\ & \left. + \frac{1}{\sqrt{N(N-1)(N-2)(N-3)}} (a_2F)_N(g) \right\} \\ & + 4 \left\{ \sqrt{(N+1)(N+2)} (a_{-1}F)_N(g) + \frac{1}{\sqrt{N(N-1)}} (a_1F)_N(g) \right\} \\ & + 6(a_0F)_N(g), \end{aligned}$$

or, more abstractly:

$$\frac{d}{dg} F(g) = BF(g), \quad F(0) = 1.$$

$$\begin{aligned} \text{II. } F_N = 4g \left\{ \frac{\sqrt{(N+1)N}}{N} (a_{-1}F)_N + \frac{3}{N} (a_0F)_N + \frac{3}{N\sqrt{(N-1)(N-2)}} (a_1F)_N \right\} \\ + 4g \frac{1}{N\sqrt{(N-1)(N-2)(N-3)(N-4)}} (a_2F)_N + F_4^0 \delta_{N4}, \end{aligned}$$

or, more abstractly:

$$F = AF + F^0, \quad F^0 = \delta_{N4} \frac{4!}{\sqrt{3!}} g(2\pi)^4 \delta(p_1 + \dots + p_4).$$

Operations (6), just like the operations A and B , have no operator meaning, but generate unbounded bilinear forms. Let us dwell on the properties of these bilinear forms.

Lemma 1. For infinitely differentiable functions f_N and g_N with compact support, the relations

$$(N-2+s)! (g_N, (a_{-s}f)_N) = (N-2-s)! ((a_s g)_N, f_N), \quad s = \pm 2, \pm 1, 0$$

hold.

Proof follows from the definition of the operations a_s .

Lemma 2. If the components of the columns f and g satisfy the condition of Lemma 1, then the bilinear form (g, Bf) is symmetric.

Remark. The bilinear form (g, Af) is nonsymmetric because of the absence of the operation a_{-2} .

4. Let us smooth our equations and operations (6), introducing into them a form factor by means of the substitution

$$\delta(p_1 + \dots + p_4) \rightarrow \varphi(p_1, \dots, p_4) \in H_4, \quad \text{Im } \varphi = 0,$$

$$\varphi(p_1, p_2, p_3, p_4) = \varphi(-p_1, -p_2, -p_3, -p_4).$$

Thus we have $a_s \rightarrow a_s(\varphi)$, $B \rightarrow B(\varphi)$. Such operators are already well defined. The following convinces us of this.

Lemma 3. The estimates

$$\|(a_s(\varphi)f)_N\| \leq \frac{N!}{(N-2-s)!} \|\varphi\| \|f_{N-2s}\|, \quad s = \pm 2, \pm 1, 0$$

hold.

Proof follows from the definition of the operators $a_s(\varphi)$ and Schwarz's inequality.

Corollary. The operator $B(\varphi)$ is unbounded. It is an operator-valued Jacobi matrix (7) and is defined on the everywhere dense set in H of finite columns.

Lemma 4. The following conjugation properties hold:

$$(N-2+s)!(g_N, (a_{-s}(\varphi)f)_N) = (N-2-s)!((a_s(\varphi)g)_N, f_N),$$

$$s = \pm 2, \pm 1, 0.$$

Lemma 5. The operator $B(\varphi)$ is symmetric on the set of finite columns.

Corollary. The operator $B(\varphi)$ is real. Therefore, by Lemma 5, it admits a self-adjoint extension.

Denote the smoothed equation I by I_φ . Then the following holds.

Theorem 1. A solution of equation I_φ exists in H and is given by the equality

$$F(g) = \exp(gB'(\varphi))F(0), \quad \text{Re } g = 0, \quad F(0) = 1,$$

where $B'(\varphi)$ is some self-adjoint extension of the operator $B(\varphi)$.

Proof follows from the theorem on the representation of functions of an unbounded self-adjoint operator.

5. Equations (4) possess one remarkable property: the operator B that defines them is a symmetric Jacobi operator-valued matrix. Let us recall in this connection that the Lippmann–Schwinger and Bethe–Salpeter equations in the Euclidean region are likewise defined by a symmetric operator.

By contrast, the operator A defining equations (5) (they can also be obtained from Schwinger equations^{8–10}) is not symmetric. Equations (4), moreover, are more meaningful physically: they completely restore the entire structure of the S -matrix. It seems to us that first one must solve equations (4) and normalize the solutions to the vacuum multipliers.

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