

ON THE BOUNDEDNESS OF SOLUTIONS OF CERTAIN SYSTEMS OF LINEAR EQUATIONS IN PARTIAL DIFFERENCES

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Abstract

Full Text

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MATHEMATICS

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ON THE BOUNDEDNESS OF SOLUTIONS OF CERTAIN SYSTEMS OF LINEAR EQUATIONS IN PARTIAL DIFFERENCES

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We consider the boundary-value problem:

$$\Delta_1 y(t_1, t_2) - \sum_{j=0}^k P_j \Delta_2^j y(t_1, t_2) = f(t_1, t_2),$$

$$y|_{0 \leq t_1 < \delta_1} = g(t_1, t_2). \quad (1)$$

Here $y(t_1, t_2)$, $f(t_1, t_2)$, $g(t_1, t_2)$ are vector functions with values belonging to some complex Banach space E , given in the domain $0 \leq t_1, t_2 < \infty$ and bounded in every domain $0 \leq t_i \leq a_i < \infty$, $i = 1, 2$. The coefficients of the equation P_j ($j = 0, 1, 2, \dots, k$) are linear operators acting in E . In contrast to ^(5, 6), we do not require pairwise commutativity of these operators.

Δ_1, Δ_2 are difference operators:

$$\Delta_1 y(t_1, t_2) = [y(t_1 + \delta_1, t_2) - y(t_1, t_2)] / \delta_1,$$

$$\Delta_2 y(t_1, t_2) = [y(t_1, t_2 + \delta_2) - y(t_1, t_2)] / \delta_2.$$

$$(\delta_1, \delta_2 > 0)$$

The system (1) does not belong to the class of equations considered by M. A. Rutman in ⁽³⁾, since it has no leading term.

In the present paper, necessary and sufficient conditions are obtained under which to every pair of vector functions $f(t_1, t_2)$ and $g(t_1, t_2)$, bounded in the domain $0 \leq t_1, t_2 < \infty$, there corresponds a bounded solution $y(t_1, t_2)$ of problem (1). Such a problem will be called stable.

Consider the family of operators $\sum_{j=0}^k P_j \alpha^j$, where α ranges over the closed disk $|z + 1/\delta_2| \leq 1/\delta_2$. Let Π_α be the set of spectral points of this family. (By the spectrum of an operator A we mean the set of points λ at which $A - \lambda I$ has no bounded inverse.)

Theorem 1. *In order that problem (1) be stable, it is necessary and sufficient that the set Π_α lie inside the disk $|z + 1/\delta_1| < 1/\delta_1$.*

Without loss of generality, we shall consider problem (1) with zero initial conditions. We transform system (1) into the form

$$\Delta_1 y(t_1, t_2) - \sum_{j=0}^k Q_j y(t_1, t_2 + j\delta_2) = f(t_1, t_2),$$

$$y|_{0 \leq t_1 < \delta_1} = 0, \quad (2)$$

where the coefficients Q_j are related to P_j by certain linear relations. Taking these relations into account, we obtain the following equivalent formulation of the theorem.

Problem (2) is stable if and only if the set of spectral points of the family of operators $\sum_{j=0}^k Q_j \alpha^j$, where α runs over the closed unit circle, lies inside the circle $|z + 1/\delta_1| < 1/\delta_1$.

It is easy to show that system (2) is equivalent to the equation

$$y(t_1, t_2) - \sum_{j=0}^k S_1 Q_j y(t_1, t_2 + j\delta_2) = S_1 f(t_1, t_2),$$

or

$$y(t_1, t_2) - S_1 A y(t_1, t_2) = S_1 f(t_1, t_2). \quad (3)$$

The vector-functions $y(t_1, t_2)$ and $f(t_1, t_2)$ on the grid $\{t_1 = n\delta_1; t_2 = m\delta_2; n, m = 0, 1, 2, \dots\}$ define infinite matrices whose entries are vectors from E .

The operator A on the indicated grid is realized by the infinite matrix:

$$A = \begin{pmatrix} Q_0 & Q_1 & \cdot & \cdot & \cdot & Q_k & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & Q_0 & Q_1 & \cdot & \cdot & \cdot & Q_{k-1} & Q_k & 0 & \cdot & \cdot \\ 0 & 0 & Q_0 & Q_1 & \cdot & \cdot & \cdot & Q_{k-1} & Q_k & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix},$$

where Q_j ($j = 0, 1, 2, \dots, k$) are operators acting in E .

S_1 is the operator inverse to the difference operator Δ_1 . On the grid under consideration it is realized by multiplying the matrix $\{f(n\delta_1, m\delta_2)\}_{n,m=0}^\infty$ on the right by the infinite matrix

$$\begin{pmatrix} 0 & \delta_1 I & \delta_1 I & \cdots \\ 0 & 0 & \delta_1 I & \cdots \\ 0 & 0 & 0 & \cdots \end{pmatrix}$$

(0 and I are, respectively, the zero and identity operators acting in the space E).

The solution of equation (3) can be represented in the form:

$$y = -(2\pi i)^{-1} \oint_{\gamma} (I - \lambda S_1)^{-1} (A - \lambda I)^{-1} S_1 f d\lambda. \quad (4)$$

(see (1)). Here γ is an arbitrary contour surrounding the spectrum of the operator A .

In what follows the following lemma is used.

Lemma. Every spectral point of the operator A belongs to the spectrum of the operator

$$\sum_{j=0}^k Q_j \alpha^j$$

for some $|\alpha| \leq 1$.

We give the scheme of the proof of the theorem.

Sufficiency. The condition of the theorem makes it possible to choose the contour γ so that it lies entirely inside the circle $|z + 1/\delta_1| < 1/\delta_1$. The operator $(I - \lambda S_1)^{-1} S_1$ is represented in the form

$$(I - \lambda S_1)^{-1} S_1 = \begin{pmatrix} 0 & \delta_1 I & \delta_1(1 + \lambda\delta_1)I & \delta_1(1 + \lambda\delta_1)^2 I & \delta_1(1 + \lambda\delta_1)^3 I & \cdots \\ 0 & 0 & \delta_1 I & \delta_1(1 + \lambda\delta_1)I & \delta_1(1 + \lambda\delta_1)^2 I & \cdots \\ 0 & 0 & 0 & \delta_1 I & \delta_1(1 + \lambda\delta_1)I & \cdots \end{pmatrix}.$$

It follows from this that this operator is uniformly bounded on the contour γ . Sufficiency is easily obtained from formula (4).

Necessity. Suppose first that for some α_0 ($|\alpha_0| \leq 1$) the operator

$$\sum_{j=0}^k Q_j \alpha_0^j$$

has a spectral point λ_0 lying outside

of the circle $|z + 1/\delta_1| < 1/\delta_1$. For simplicity we restrict ourselves to the case when λ_0 is an eigenvalue and g_0 is the corresponding eigenvector:

$$\sum_{j=0}^k Q_j \alpha_0^j g_0 = \lambda_0 g_0.$$

As the right-hand side of equation (2) we take the vector-function which, on the grid $\{t_1 = n\delta_1, t_2 = m\delta_2, n, m = 0, 1, 2, \dots\}$, defines the matrix

$$\begin{pmatrix} g_0 & g_0 & g_0 & \dots \\ \alpha_0 g_0 & \alpha_0 g_0 & \alpha_0 g_0 & \dots \\ \alpha_0^2 g_0 & \alpha_0^2 g_0 & \alpha_0^2 g_0 & \dots \end{pmatrix}.$$

Then $(\lambda_0 I - A)f_0 = 0$ and

$$(\lambda I - A)f_0 = (\lambda - \lambda_0)f_0.$$

Hence

$$(\lambda I - A)^{-1} f_0 = \frac{f_0}{\lambda - \lambda_0}.$$

Substituting this expression into the solution formula (4), we obtain

$$y = (I - \lambda_0 S_1)^{-1} S_1 f_0.$$

From this the unboundedness of the solution y follows immediately.

If λ_0 lies on the boundary of the circle $|z + 1/\delta_1| < 1/\delta_1$, then the right-hand side must be given by the matrix

$$\begin{pmatrix} g_0 & g_0 e^{i\varphi} & g_0 e^{2i\varphi} & \dots \\ \alpha_0 g_0 & \alpha_0 g_0 e^{i\varphi} & \alpha_0 g_0 e^{2i\varphi} & \dots \\ \alpha_0^2 g_0 & \alpha_0^2 g_0 e^{i\varphi} & \alpha_0^2 g_0 e^{2i\varphi} & \dots \end{pmatrix},$$

where $e^{i\varphi} = 1 + \lambda_0 \delta_1$.

Using the methods developed in (2-4), one can extend the result obtained to equations with variable coefficients. Consider the boundary-value problem

$$\Delta_1 y(t_1, t_2) - \sum_{j=0}^k P_j(t_1, t_2) \Delta_2^j y(t_1, t_2) = f(t_1, t_2), \quad (5)$$

$$y|_{0 \leq t_1 < \delta_1} = g(t_1, t_2).$$

We shall assume that the operator-coefficients $P_j(t_1, t_2)$ satisfy the following conditions:

1. The operator-functions $P_j(t_1, t_2)$ ($j = 0, 1, 2, \dots, k$) are continuous throughout the entire domain of variation of the independent variables.
2. All families $P_j(t_1, t_2)$ are compact.
3. The $P_j(t_1, t_2)$ have weak variation at t_1 -infinity.

This means that for every $\varepsilon > 0$ there exists $T = T(\varepsilon)$ such that from the conditions

$$t'_1 > T, \quad t''_1 > T, \quad |t'_1 - t''_1| + |t'_2 - t''_2| \leq 1$$

it follows that

$$\|P_j(t'_1, t'_2) - P_j(t''_1, t''_2)\| < \varepsilon.$$

Consider all possible ω_{t_1} -limit operators $P_j^{(\omega_{t_1})}$, generated by sequences of points $(t_1^{(n)}, t_2^{(n)})$, common for all $P_j(t_1, t_2)$ ($j = 0, 1, 2, \dots, k$), with $t_1^{(n)} \rightarrow \infty$. Form the family of operators

$$\sum_{j=0}^k P_j^{(\omega_{t_1})} \alpha^j,$$

where α ranges over the closed circle $|z + 1/\delta_2| \leq 1/\delta_2$. Let Ω_α be the set of points of the spectrum of this family.

Theorem 2. In order that, in problem (5), every bounded vector-functions $f(t_1, t_2)$ and $g(t_1, t_2)$ correspond to a bounded solution $y(t_1, t_2)$, it is necessary and sufficient that the set Ω_α lie inside the circle $|z + 1/\delta_1| < 1/\delta_1$.

The results presented are applicable, in particular, to the problems

$$\Delta_1 y(t_1, t_2) - a \Delta_2 y(t_1, t_2) - A(t_1, t_2) y(t_1, t_2) = f(t_1, t_2),$$

$$y|_{0 \leq t_1 < \delta_1} = g(t_1, t_2);$$

$$\Delta_1 y(t_1, t_2) - a^2 \Delta_2^2 y(t_1, t_2) - A(t_1, t_2) y(t_1, t_2) = f(t_1, t_2),$$

$$y|_{0 \leq t_1 < \delta_1} = g(t_1, t_2)$$

(a is a real number), which are difference analogues of the differential problems

$$\partial u / \partial t - a \partial u / \partial x - A(x, t)u = f(x, t),$$

$$u(x, 0) = \varphi(x);$$

$$\partial u / \partial t - a^2 \partial^2 u / \partial x^2 - A(x, t)u = f(x, t),$$

$$u(x, 0) = \varphi(x),$$

considered by E. Ya. Melamed in (4). The stability criteria established by him for these problems can be obtained from the last theorem as limiting cases.

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