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Abstract

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MATHEMATICS

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ON TOPOLOGIES OF GENERALIZED INTERVALS

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§ 1. By an interval, an ordered set, and a mapping we shall mean, respectively, a generalized continuum interval $(^3)^*$, a linearly ordered set, and a continuous mapping. A Π -topology on an ordered set C is a topology under which C is an interval, one of the orders induced by connectedness $(^3)$ in which coincides with the given one.

Lemma 1. Let (A, B) be such a cut in an ordered set C that A (B) has no greatest (least) element. In order that the topology on C be a Π -topology, it is necessary that A (B) be an open set in this topology.

Theorem 1. If in an ordered set C there are gaps, then it is impossible to introduce a Π -topology on C .

Theorem 2. If for a point $a \in \vec{C}$ every set of the form $]a_1, a[$ ($]a, a_1[$), where $a_1 < a$ ($a_1 > a$), is infinite, then $\{a\}$ is a closed set in C .

Theorem 3. If in an ordered set C there are neither gaps nor jumps, then every Π -topology** on C majorizes the interval topology; hence it is a T_2 -topology.

Theorem 4. If there exists a partition of an ordered set C into three nonempty subsets A_1, A_2, A_3 such that $a_1 < a_2 < a_3$ (a_1, a_2, a_3 are arbitrary elements of the sets A_1, A_2, A_3 , respectively), and A_1 has no greatest element, A_3 has no least element, while the set A_2 consists of a finite and, moreover, even number of elements, then it is impossible to introduce a Π -topology on C .

An element a of a subset U of an ordered set C will be called an end element of the set U in C if in C there exists an interval of one of the forms $[a_1, a[$ or $]a, a_1]$, whose intersection with U is the empty set.

Theorem 5. If a is an end element of a set U open (closed) in the interval $\vec{C}$, then the set $\{a\}$ is open (closed) in $\vec{C}$.

Corollary 1. If $\{a\}$ is a closed (open) set in $\vec{C}$ and there exists a point $a_1 \in \vec{C}$ such that $]a_1, a[= \emptyset$ or $]a, a_1[= \emptyset$, then every set U open (closed) in $\vec{C}$, containing a , also contains a_1 .

Corollary 2. If a point a is not an end of the interval $\vec{C}$ and $\{a\}$ is open in $\vec{C}$, then in $\vec{C}$ there exist points a_1 and a_2 such that $]a_1, a_2[= \{a\}$; the closure of $\{a\}$ in this case is $[a_1, a_2]$. If a is the first (last) element of $\vec{C}$ and $\{a\}$ is an open set in $\vec{C}$, then in $\vec{C}$ there exists a point a_1 such that $] \leftarrow, a_1[= \{a\}$ ($]a_1, \rightarrow [= \{a\}$); the clos—

* In the definition of an interval in ⁽³⁾ one should add the item: 3. If the connected component V_a of the space $A - \{a\}$ contains the connected component V_b of the space $A - \{b\}$ or an end a_i of the interval A , then the connected component V_c of the space $A - \{c\}$, containing exactly V_b or a_i , is either itself contained in V_a , or contains V_a ($a \in A$, $b \in A$, $c \in A$).

** The existence of a Π -topology follows from ⁽³⁾.

tion $\{a\}$ in this case is equal to $[a, a_1] \setminus ([a_1, a])$ (the existence of each of the points a_1, a_2 follows from Theorem 2).

Theorem 6. In order that a Π -topology can be introduced on an ordered set C , it is necessary and sufficient that there be no gaps in C and that there exist no partition of the set C satisfying the condition of Theorem 4.

We shall prove sufficiency. For each $a \in C$ there are only two possibilities: I. in $] \leftarrow, a[$ there is no greatest element, or in $]a, \rightarrow [$ there is no least element. II. In $] \leftarrow, a[$ there is a greatest element and in $]a, \rightarrow [$ there is a least element.

In case I, a is a limit element; in case II, a is an isolated element of the set C . We divide the set of all isolated elements into equivalence classes, assigning to one equivalence class all and only those elements between any two of which there is not a single limit element. Choose one representative a from each equivalence class. There are only the following possibilities for a : 1) there exists a limit element a' such that the number of elements between a and a' is finite; 2) there exists a sequence

$$\dots < a_{-n} < \dots < a_{-2} < a_{-1} = a = a_1 < a_2 < \dots < a_n < \dots$$

such that

$$]a_i, a_j[= \bigcup_{i < k < j} a_k$$

$$(n = 1, 2, \dots, < \omega);$$

3) there exists a sequence

$$a_{-m} < a_{-m+1} < \dots < a_{-1} = a = a_1 < a_2 < \dots < a_n < \dots$$

such that

$$] \leftarrow, a[= \bigcup_{i=1}^m a_{-i}$$

and

$$]a_i, a_j[= \bigcup_{i < k < j} a_k$$

$$(n = 1, 2, \dots, < \omega);$$

4) there exists a sequence

$$\dots < a_{-n} < \dots < a_{-2} < a_{-1} = a = a_1 < a_2 < \dots < a_m$$

such that

$$[a, \rightarrow [= \bigcup_{i=1}^m a_i$$

and

$$]a_i, a_j[= \bigcup_{i < k < j} a_k$$

$$(n = 1, 2, \dots, < \omega);$$

5) there exists a sequence

$$a_{-m} < \dots < a_{-1} = a = a_1 < a_2 < \dots < a_n$$

such that

$$C = \bigcup_{-m < i < n} a_i.$$

We introduce a topology in C . We declare the following sets open in C : $] \leftarrow, a[$ and $]a, \rightarrow [$ in cases I and 1), if the number of elements between a and a' is odd; $] \leftarrow, a[$ and $[a, \rightarrow [$ in case 1), if the number of elements between a and a' is even; in cases 2)–5), either sets of the form $] \leftarrow, a_{\pm(2k-1)}[$ and $]a_{\pm(2k-1)}, \rightarrow [$, or sets of the form $] \leftarrow, a_{\pm 2k}[$ and $]a_{\pm 2k}, \rightarrow [$, indifferently which ($k = 1, 2, \dots$).

The open sets described are taken as a subbase of a topology, which we shall call interval-alternating (i.a.). From the definition of the i.a. topology it follows that for each point a of C the sets $] \leftarrow, a[$ and $]a, \rightarrow [$ are either both open or both closed in C . Hence, if $C_1 \subset C$ and in C there exist points $a_1 < a_2 < a_3$ such that a_1 and $a_3 \in C_1$, $a_2 \notin C_1$, then the nonempty sets $] \leftarrow, a_2[\cap C_1$ and $]a_2, \rightarrow [\cap C_1$ are either both open or both closed in C_1 , which indicates that C_1 is disconnected. We shall show that a subspace C_2 of the space C is connected if, for any points a_1 and a_2 of C_2 ($a_1 \neq a_2$), every point of C lying between them belongs to C_2 . Let U be a nontrivial open-and-closed subset of C_2 . From the definition of the i.a. topology it follows that every closed nontrivial subset U in C_2 has at least one end element a_0 in C_2 ; moreover, if a_0 is an end element of a subset closed (open) in C_2 , then $\{a_0\}$ is closed (open) in C_2 . Hence, by supposition, there exists an end element a_0 of the set U in C_2 , and $\{a_0\}$ is an open-and-closed set in C_2 . Consequently, there exist sets U_1 , closed, and U_2 , open in C , such that $U_i \cap C_2 = \{a_0\}$ ($i = 1, 2$). It is obvious that a_0 is an end element of U_i in C ($i = 1, 2$). Hence $\{a_0\}$ is an open-and-closed set in C , which

contradicts the definition of the i.a. topology. Consequently, C_2 is connected, and from Proposition 1 in (3) we conclude that the i.a. topology is a Π -topology.

Corollary 1. Every i.a. topology is uniquely determined by the set of all one-sided open sets in this topology.

Corollary 2. For every Π -topology on an ordered set C there exists a unique i.c. topology majorized by it.*

Corollary 3. The interval $[a_1, a_2]$ with the i.c. topology is a quasicompact space ⁽¹⁾.

This follows from Alexandroff's theorem in ⁽²⁾.

Corollary 4. If an ordered set C can be endowed with a quasicompact Π -topology, then such a topology on C , generally speaking, is not unique, and any two such topologies on C are incomparable ⁽¹⁾.

Theorem 7. In order that the topology of the interval $\vec{C}$ be an i.c. topology, it is necessary and sufficient that the connected components of every set U open in $\vec{C}$ be sets open in $\vec{C}$.

Theorem 8. In order that the topology of the interval $\vec{C}$ be an i.c. topology, it is necessary and sufficient that, for every $a \in \vec{C}$ such that $\{a\}$ is a closed set in $\vec{C}$, and for every set U open in $\vec{C}$ and containing a , one of the following conditions hold: 1) if a is not an end of the interval $\vec{C}$, then in $\vec{C}$ there exists an interval $[a_1, a_2]$ ($a_1 < a < a_2$) such that $[a_1, a_2] \subset U$; 2) if a is the first (last) element of $\vec{C}$, then in $\vec{C}$ there exists an interval $] \leftarrow, a_1]$ ($[a_2, \rightarrow [$), where $a_2 < a < a_1$, such that $] \leftarrow, a_1] \subset U$ ($[a_2, \rightarrow [\subset U$).

Theorem 9. In order that a topology majorizing some i.c. topology in an ordered set C be a Π -topology, it is necessary and sufficient that the sets U open in this topology satisfy the following two conditions: 1. In U there is no end element a_0 such that $\{a_0\}$ is a closed set in C . 2. The nonempty complement** of the nonempty intersection of the set U with an arbitrary interval to the whole interval is not a set open in this interval.

Theorem 10. If in an ordered set C there is at least one limit point a_0 and there exists at least one Π -topology on C , then on C there exist infinitely many Π -topologies that are not i.c. topologies.

Theorem 11. In order that a connected topological space, distinct from the antidiscrete two-point set, be an interval, it is necessary and sufficient that there exist a condensation of it onto some interval endowed with an i.c. topology.

We shall call a topological space X orderable if on it there exists a transitive relation $<$, called an order, satisfying the following conditions: 1. If $x, y \in X$, then one and only one of the relations $x < y$, $x = y$, $y < x$ holds. 2. Each connected component A of the space X is an interval $\bar{A}$, and the order induced by the relation $<$ on $\bar{A}$ coincides with one of the orders induced by the connectedness of the interval $\bar{A}$. 3. If x, y belong to distinct connected

components $A(x)$ and $A(y)$ of the space X and $x < y$, then $x' < y'$ for all $x' \in A(x)$ and $y' \in A(y)$.

Theorem 12. If the space X is orderable by a relation $<$ in such a way that, under this ordering, in the set of points of the space X there are neither gaps nor jumps, then for X there exists an interval $\bar{A}$, endowed with an interval topology, such that X condenses onto $\bar{A}$. If the topological space X condenses onto some interval $\bar{A}$, then X is an orderable space.

Theorem 13. In order that every Π -topology in an ordered set C be an i.c. topology, it is necessary and sufficient that every cut in C be a jump (except for the antidiscrete two-point set).

§ 2. We introduce the notion of the complete connected oriented union of arbitrary non-one-point intervals distinct from the antidiscrete

* Except for the case when $\vec{C}$ is the antidiscrete two-point set ⁽²⁾.

** In item 3b) in Theorem 2 in ⁽³⁾ one should also take only the nonempty complement.

semicolons. Take a family of segments $\{\vec{A}_\lambda\}$, where λ ranges over the ordered set Λ . To each point $a \in \vec{A}_\lambda$ assign the pair (λ, a) . Order the set of all pairs by the lexicographic rule. Identify each pair of points (λ_1, a_1) and (λ_2, a_2) such that $](\lambda_1, a_1); (\lambda_2, a_2)[= \emptyset$, in the following cases: 1) each of the sets $\{a_1\}$, $\{a_2\}$ is closed respectively in $\vec{A}_{\lambda_1}$, $\vec{A}_{\lambda_2}$; 2) each of the sets $\{a_1\}$, $\{a_2\}$ is open respectively in $\vec{A}_{\lambda_1}$, $\vec{A}_{\lambda_2}$. Denote the resulting set by C' .

Consider the set of all cuts (A, B) in the set C' . If in A (B) there is no greatest (least) element, while in B (A) there is a least (greatest) element (λ, a) such that $\{a\}$ is an open subset of $\vec{A}_\lambda$, or the cut (A, B) is a gap, then add to C' an element α , which we place immediately after every element of A and before every element of B . Denote the resulting ordered set by C . Obviously, for each ordered set $\vec{A}_\lambda$ there exists a natural similarity correspondence $f_\lambda : A_\lambda \rightarrow C_\lambda \subset C$. For each $\lambda \in \Lambda$, declare open in C' the image under f_λ of every set open in $\vec{A}_\lambda$ that contains no endpoint a_λ of the segment $\vec{A}_\lambda$ such that $\{a_\lambda\}$ is a closed set in $\vec{A}_\lambda$ and $f_\lambda(a_\lambda)$ is neither the first nor the last element of the set C ; also declare open all sets open in the following o.t. topology in C , in which the open one-point sets in C are precisely those one-point sets each of which is the image under some f_λ of a set open in $\vec{A}_\lambda$, and all sets of the form $U_1 \cup U_2$, where U_1 is the image under some f_λ of a set open in $\vec{A}_\lambda$ containing such an endpoint a_λ of the segment $\vec{A}_\lambda$ that $\{a_\lambda\}$ is a closed set in $\vec{A}_\lambda$, and U_2 is a set open in the above-described o.t. topology in C such that a_λ is an endpoint of $C' - U_2$ in C and $C_\lambda \subset C - U_2$. We take all the listed open sets as a prebase for the topology in C .* From Theorem 9 and Corollaries 1 and 2 to Theorem 6 it follows that a P-topology has been obtained. The segment $\vec{C}$ constructed in this way will be called the complete connected oriented union of the segments $\vec{A}_\lambda$, and we shall

write

$$\vec{C} = \overline{\bigcup_{\lambda \in \Lambda} \vec{A}_\lambda}.$$

The topology constructed is the weakest P-topology in C for which every f_λ ($\lambda \in \Lambda$) is a homeomorphism.

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REFERENCES CITED

- ¹ N. Bourbaki, *General Topology*, Moscow, 1968.
- ² John L. Kelley, *General Topology*, Moscow, 1968.
- ³ E. D. Khalimskii, *Dokl. Akad. Nauk SSSR* 185, No. 2 (1969).**

* The topology in $\overline{\cup \vec{A}_\lambda}$ in (3) should be introduced in the same way as here.

** The following corrections must be made in paper (3).

1. Two mappings $f : \dot{e}_n \times \vec{A} \rightarrow X$ and $\bar{f} : \dot{e}_n \times \bar{A} \rightarrow X$ from the class of mappings under consideration should be regarded as mutually opposite if $f(x) = \bar{f}(x)$ ($x \in \dot{e}_n \times \vec{A}$; $\bar{A}$ differs from $\vec{A}$ only by orientation).
2. In Proposition 3, instead of the words "neither one nor $\{e\}$ equivalence classes," one should read, respectively, "neither one, e ." In the footnote to Proposition 3, instead of $\{f\}$ and $\{f_i\} \neq \{e\}$, one should read, respectively, f and $f_i \neq e$.

Note: Figure translations are in progress. See original paper for figures.

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