

ON THE BEST APPROXIMATION OF CONTINUOUS FUNCTIONS IN THE METRIC $\|(C[a,b]\|)$ BY GENERALIZED POLYNOMIALS

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Abstract

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MATHEMATICS

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ON THE BEST APPROXIMATION OF CONTINUOUS FUNCTIONS IN THE METRIC $C[a, b]$ BY GENERALIZED POLYNOMIALS

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In the present paper we indicate necessary and sufficient conditions that must be imposed on a system of functions $\{\varphi_i(x)\}_1^{n+1}$ so that every polynomial

$$P(c, x) = \sum_{i=1}^{n+1} c_i \varphi_i(x)$$

least deviating in the metric $C[a, b]$ from a function $f(x) \in C[a, b]$ possess no fewer than $k + 1$ ($0 \leq k \leq n + 1$) points of alternance.

Definition. A system of continuous and linearly independent functions $\{\varphi_i(x)\}_1^{n+1}$ on $[a, b]$ will be called a **system of type k** ($0 \leq k \leq n$) if: 1) for any points $a \leq x_1 < x_2 < \dots < x_{n+1} \leq b$ such that

$$\begin{vmatrix} \varphi_1(x_1) & \dots & \varphi_{n+1}(x_1) \\ \dots & \dots & \dots \\ \varphi_1(x_{n+1}) & \dots & \varphi_{n+1}(x_{n+1}) \end{vmatrix} = 0,$$

there exist $i + 1$ rows ($k \leq i \leq n$) for which all minors of order $i + 1$ are equal to zero (at least for one i); 2) in at least one such determinant of order $i + 1$ all complementary minors D_j^i ($j = 1, \dots, i + 1$) of the elements of at least one column are different from zero, and in the sequence $D_1^i, \dots, D_{i+1}^i$ there are no more than $i - k$ sign changes; moreover, the set of points $\{x_i\}_1^{n+1}$ for which in this sequence (for some i) there will be exactly $i - k$ sign changes.

Examples of systems of type k :

1. Define on the segment $[-1, 2]$ continuous functions in the following way:

$$\varphi_1(x) = 1, \quad \varphi_2(x) = x, \dots, \quad \varphi_{k-1}(x) = x^{k-2}, \quad \varphi_k(x) = x^k,$$

$$\varphi_{k+1}(x) = x^{k+1} + a_0x^k + \dots + a_k,$$

$$\varphi_i(x) = \begin{cases} 0 & \text{for } x \in [-1, 1], \\ x^i - 1 & \text{for } x \in (1, 2], \end{cases} \quad (i = k+2, \dots, n)$$

$$\varphi_{n+1}(x) = \begin{cases} 0 & \text{for } x \in [-1, 1], \\ x^{k-1} - 1 & \text{for } x \in (1, 2]. \end{cases}$$

It is possible to choose the parameters $a_0, a_1, \dots, a_k$ so that no k functions from $\{\varphi_i(x)\}_1^{n+1}$ form a Chebyshev system (T -system) on $[-1, 2]$, but the whole system will be of type k .

2. The system of functions $1, x^2, x^4, \dots, x^{2n}$ will be a system of type 1 on the segment $[-a, a]$, where $a > 0$.

Remark. If $\{\varphi_i(x)\}_1^{n+1}$ form a Chebyshev system (T -system) on $[a, b]$, then we shall regard its type as equal to $n+1$.

Theorem. In order that, for every function $f(x)$ continuous on $[a, b]$, every polynomial least deviating from it

$$p(c, x) = \sum_{i=1}^{n+1} c_i \varphi_i(x)$$

possess no fewer than $k+1$ point alternations, and, in order that for every function $f(x)$ at least one of the polynomials of least deviation from it have exactly $k+1$ point alternations, it is necessary and sufficient that the system $\{\varphi_i(x)\}_1^{n+1}$ be of type k ($0 \leq k \leq n+1$) on $[a, b]$.

In what follows we shall use the criterion for polynomials of least deviation when, on $[a, b]$, the functions $\{\varphi_i(x)\}_1^{n+1}$ do not form a Chebyshev system $(1,2)$, and the following lemmas.

Lemma 1. The equality is valid

$$\begin{aligned} \sum_{i=k}^{k+l-1} (-1)^{k+l-1} D \left(\begin{matrix} \varphi_1, \dots, \varphi_k \\ x_1, \dots, x_{k-1}, x_i \end{matrix} \right) D \left(\begin{matrix} \varphi_1, \dots, \varphi_k \\ x_l, \dots, x_{i-1}, x_{i+1}, \dots, x_{k+l-1}, x \end{matrix} \right) = \\ = D \left(\begin{matrix} \varphi_1, \dots, \varphi_k \\ x_1, \dots, x_{k-1}, x \end{matrix} \right) D \left(\begin{matrix} \varphi_1, \dots, \varphi_k \\ x_l, \dots, x_{k+l-1} \end{matrix} \right) \quad (k \geq 1, 1 \leq l \leq k), \end{aligned}$$

where

$$D \left(\begin{matrix} \varphi_1, \dots, \varphi_k \\ x_{t_1}, \dots, x_{t_k} \end{matrix} \right) = \begin{vmatrix} \varphi_1(x_{t_1}), \dots, \varphi_k(x_{t_1}) \\ \dots \\ \varphi_1(x_{t_k}), \dots, \varphi_k(x_{t_k}) \end{vmatrix}.$$

Lemma 2. If

$$D \begin{pmatrix} \varphi_{t_1}, \dots, \varphi_{t_i} \\ x_1, \dots, x_i \end{pmatrix} \neq 0,$$

then

$$\begin{aligned} & (-1)^{n-i+1} D \begin{pmatrix} \varphi_1, \dots, \varphi_{n+1} \\ x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_{n+1}, x \end{pmatrix} + \\ & + \sum_{r=i+1}^{n+1} (-1)^{n-r+1} \frac{D \begin{pmatrix} \varphi_{t_1}, \dots, \varphi_{t_i} \\ x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_i, x_r \end{pmatrix}}{D \begin{pmatrix} \varphi_{t_1}, \dots, \varphi_{t_i} \\ x_1, \dots, x_i \end{pmatrix}} \times \\ & \times D \begin{pmatrix} \varphi_1, \dots, \varphi_{n+1} \\ x_1, \dots, x_{r-1}, x_{r+1}, \dots, x_{n+1}, x \end{pmatrix} = \\ & = \frac{D \begin{pmatrix} \varphi_{t_1}, \dots, \varphi_{t_i} \\ x_1, \dots, x_{j-1}, x_{j+1}, \dots, x_i, x \end{pmatrix}}{D \begin{pmatrix} \varphi_{t_1}, \dots, \varphi_{t_i} \\ x_1, \dots, x_i \end{pmatrix}} D \begin{pmatrix} \varphi_1, \dots, \varphi_{n+1} \\ x_1, \dots, x_{n+1} \end{pmatrix} \\ & (j = 1, \dots, i). \end{aligned}$$

Lemma 3. If the system $\{\varphi_i(x)\}_1^{n+1}$ is of type k , then in the sequence $D_1^{n+1}, D_2^{n+1}, \dots, D_{n+2}^{n+1}$ ($D_j^{n+1} \neq 0$; $j = 1, \dots, n+2$) there will be no more than $n+1-k$ changes of sign, where D_j^{n+1} ($j = 1, \dots, n+2$) are obtained by deleting the j -th row from the matrix

$$\left\| \begin{array}{c} \varphi_1(x_1) \dots \varphi_{n+1}(x_1) \\ \dots \\ \varphi_1(x_{n+2}) \dots \varphi_{n+1}(x_{n+2}) \end{array} \right\|.$$

Lemma 4. If in the sequence of complementary minors $D_1^i, \dots, D_{i+1}^i$ ($D_j^i \neq 0$, $j = 1, \dots, i+1$) of the elements of one column of some determinant of order $(i+1)$ of the matrix

$$\left\| \begin{array}{c} \varphi_1(x_{\mu_1}) \dots \varphi_{n+1}(x_{\mu_1}) \\ \dots \\ \varphi_1(x_{\mu_{i+1}}) \dots \varphi_{n+1}(x_{\mu_{i+1}}) \end{array} \right\| \quad (*)$$

($\mu_1, \mu_2, \dots, \mu_{i+1}$ are the numbers of the rows appearing in condition 1) of the definition) has p changes of sign, then in the row of nonzero complementary minors D_j^i ($j = 1, \dots, i+1$) of the elements of any other column of the same matrix $()$ there will also be p changes of sign, and if the complementary minor of at least one element of some determinant of order $(i+1)$ of the matrix $()$ is equal to zero, the complementary minors of all the remaining elements of the corresponding column are also equal to zero.

We outline the proof of the theorem.

Sufficiency. Let a system of type k and $p(c, x)$ deviate least from $f(x)$ on $[a, b]$. From (1) it follows that among the points of the minimal subset there will be $i + 1$ ($k \leq i \leq n + 1$) points $x_1, \dots, x_{i+1}$ for which: a) the matrix (*) has rank i ; b) there exist $D_j^i \neq 0$ ($j = 1, \dots, i + 1$); c) $\text{sign}[p(c, x_j) - f(x_j)] = \delta \text{sign}(-1)^j D_j^i$, where $\delta = \pm 1$.

Since in the row $D_1^i, \dots, D_{i+1}^i$ ($k \leq i \leq n + 1$) there are no more than $i - k$ changes of sign, in the row $-D_1^i, D_2^i, \dots, (-1)^{i+1} D_{i+1}^i$ there will be no fewer than k changes of sign. Consequently, at least $k + 1$ points from $\{x_i\}_1^{i+1}$ will be alternation points.

Necessity. Let every polynomial $p(c, x)$ that deviates least from $f(x)$ possess at least $k + 1$ pointwise alternations.

Suppose that the system $\{\varphi_i(x)\}_1^{n+1}$ is of type r (for definiteness, $r < k$). Then, by definition, there exists a system of points $\{x_j\}_1^{i+1}$ for which, in the corresponding row $D_1^i, \dots, D_{i+1}^i$, there will be exactly $i - r$ changes of sign.

Put $f(x_j) = \text{sign}[(-1)^j D_j^i]$ ($j = 1, \dots, i + 1$), $f(a) = \text{sign}(-D_1^i)$, $f(b) = \text{sign}(-1)^{i+1} D_{i+1}^i$, and at the remaining points of $[a, b]$ let $f(x)$ be linear. From condition 1) of the definition we have

$$\sum_{j=1}^{i+1} (-1)^j D_j^i p(c, x_j) = 0.$$

Consequently, the least deviation

$$h(c) = |p(c, x_j) - f(x_j)| = |p(c, x_j) - [\text{sign}(-1)^j D_j^i]|$$

is not less than 1. $p(c, x) \equiv 0$ deviates least from $f(x)$. For it

$$\text{sign}[p(c, x_j) - f(x_j)] = -\text{sign}[(-1)^j D_j^i]$$

and, consequently, the number of alternation points will be $r + 1 < k + 1$, which contradicts the assumption.

It is easy to show that any system of functions $\{\varphi_i(x)\}_1^{n+1}$ linearly independent on $[a, b]$ will be a system of some definite type k ($0 \leq k \leq n + 1$).

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