

ON GREEN' S FUNCTIONS FOR PROBLEMS WITH THE BESSEL DIFFERENTIAL OPERATOR

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Abstract

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MATHEMATICS

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ON GREEN' S FUNCTIONS FOR PROBLEMS WITH THE BESSEL DIFFERENTIAL OPERATOR

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Let E_{n+2} denote the $(n+2)$ -dimensional Euclidean space of points $z = (x, y, t)$, where $x = (x_1, \dots, x_n)$, $x_{n+1} = y$, $x_{n+2} = t$. Consider in this space the domain $y > 0$, $t > 0$. In this domain consider the following boundary-value problem. Find a solution of the equation

$$\mathcal{L}(D_x, B_y, D_t) = \sum_{|k|=2m} a_k D_x^{k'} B_y^{k_{n+1}} D_t^{k_{n+2}} u = 0, \quad (1)$$

where $|k| = |k'| + 2k_{n+1} + k_{n+2}$,

$$D_x^{k'} = \left(\frac{1}{i} \frac{\partial}{\partial x_1} \right)^{k_1} \cdots \left(\frac{1}{i} \frac{\partial}{\partial x_n} \right)^{k_n}, \quad D_t^{k_{n+2}} = \left(\frac{1}{i} \frac{\partial}{\partial t} \right)^{k_{n+2}},$$

$$B_y^{k_{n+1}} = \left[\left(\frac{1}{i} \right)^2 \left(\frac{\partial^2}{\partial y^2} + \frac{2\nu+1}{y} \frac{\partial}{\partial y} \right) \right]^{k_{n+1}}$$

in the domain $y > 0$, $t > 0$, satisfying the following boundary conditions:

$$H_\mu(D_x, B_y, D_t)u|_{t=0} = g_\mu(x, y) \quad (\mu = 1, 2, \dots, m). \quad (2)$$

It is assumed that the coefficients a_k of the operator $\mathcal{L}$ are constant and that the boundary operators H_μ are homogeneous with constant coefficients. In addition, it is assumed that the operator $\mathcal{L}$ is a B -elliptic operator ⁽²⁾ and that the operators $\mathcal{L}$ and H_μ satisfy the Lopatinski condition (see, for example, ⁽¹⁾). We shall seek classical solutions of this problem by means of the Fourier-Bessel transform, i.e. we shall seek solutions in the form

$$u(x, y, t) = \int_{-\infty}^{\infty} \int_0^{\infty} v(\xi, \sigma, t) e^{ix\xi} j_{\nu}(\sigma y) \sigma^{2\nu+1} d\xi d\sigma. \quad (3)$$

Then for v one obtains the boundary-value problem

$$\mathcal{L}(\xi, \sigma, D_t)v = 0, \quad H_{\mu}(\xi, \sigma, D_t)v|_{t=0} = g_{\mu}(\xi, \sigma) \quad (4)$$

on the half-line $t > 0$ for an ordinary differential equation with constant coefficients, for fixed ξ, σ . In order to single out the solutions of problem (4) that decrease exponentially as $t \rightarrow \infty$, factorization is carried out, as usual. Since the Lopatinskii condition is satisfied, problem (4) is solvable, and its solution is written in the form

$$v(\xi, \sigma, t) = \sum_{k=1}^m \tilde{v}_k(\xi, \sigma, t) \hat{g}_k(\xi, \sigma), \quad (5)$$

where

$$v_k(\xi, \sigma, t) = \frac{1}{2\pi i} \int_C \frac{N_k(\xi, \sigma, \tau)}{M^+(\xi, \sigma, \tau)} e^{i\tau t} d\tau. \quad (6)$$

is a Poisson basis; C is a contour in the upper half-plane of the complex plane, enclosing all zeros of the polynomial M^+ ($\mathcal{L} = a_0 M^+ M^-$). If in the formula for the solution of our problem we pass from the Fourier-Bessel images to the functions themselves, we obtain

$$u(x, y, t) = \sum_{j=1}^m \int_{-\infty}^{\infty} \int_0^{\infty} T_{x,y}^{\alpha,\gamma} K_j(x, y, t) g_j(\alpha, \gamma) \gamma^{2\nu+1} d\alpha d\gamma, \quad (7)$$

where the function $K_j(x, y, t)$, called the Poisson kernel, is defined by the relation

$$K_j(x, y, t) = C_{\nu} \int_{-\infty}^{\infty} \int_0^{\infty} \tilde{v}_j(\xi, \sigma, t) e^{ix\xi} j_{\nu}(\sigma y) \sigma^{2\nu+1} d\xi d\sigma. \quad (8)$$

Here and below the constants C_{ν} have a quite definite value and their particular form is important. In view of their cumbersomeness we do not write them out.

Taking into account the homogeneity of $\tilde{v}_k$ and introducing polar coordinates in the appropriate way, we obtain the following explicit formulas for the Poisson kernels. If $n + 1 + 2\nu + 1 > m_j$, where m_j is the order of the operator H_j , then

$$K_j(x, y, t) =$$

$$= C_j \int_0^\pi \sin^{2\nu} \varphi d\varphi \int_{C_{R,\delta}} d\beta \int_{|\eta|=1} \frac{\dot{\sigma}^{2\nu+1}}{(x\dot{\xi} + y\dot{\sigma} \cos \varphi + t\beta)^{n+1+2\nu+1-m_j}} \frac{N_j(\eta, \beta)}{M^+(\eta, \beta)} d\omega_\eta,$$

$$\eta = (\dot{\xi}, \dot{\sigma}). \quad (9)$$

If, however, $m_j \geq n + 1 + 2\nu + 1$, then the corresponding integral is divergent. Applying the well-known regularization device ⁽¹⁾, in this case we obtain for the Poisson kernels the following two formulas. If $2\nu + 1$ is an integer, then

$$K_j(x, y, t) =$$

$$= C_j \int_0^\pi \sin^{2\nu} \varphi d\varphi \int_{C_{R,\delta}} d\beta \int_{|\eta|=1} (x\dot{\xi} + y\dot{\sigma} + t\beta)^{m_j-n-1-2\nu-1} \times$$

$$\times \ln \frac{x\dot{\xi} + y\dot{\sigma} + t\beta}{i} \frac{N_j}{M^+} \dot{\sigma}^{2\nu+1} d\omega_\eta. \quad (10)$$

If, however, $2\nu + 1$ is a fractional number, then the formula for the Poisson kernel has the form

$$K_j(x, y, t) =$$

$$= C_j \int_0^\pi \sin^{2\nu} \varphi d\varphi \int_{C_{R,\delta}} d\beta \int_{|\eta|=1} (x\dot{\xi} + y\dot{\sigma} \cos \varphi + t\beta)^{m_j-n-1-2\nu-1} \frac{N_j}{M^+} \dot{\sigma}^{2\nu+1} d\omega_\eta.$$

The Poisson kernels obtained above can be differentiated, for $t > 0$, arbitrarily many times. When boundary operators are applied, singularities may arise. We need to have kernels such that both they themselves and all their derivatives up to a certain order are continuous in the closed half-space. This is achieved, as is known ⁽¹⁾, by constructing the so-called adjoint kernels.

Taking into account the formal self-adjointness of the generalized translation operator, with the aid of the adjoint kernels the solution is written in the form

$$u(x, y, t) =$$

$$= C \sum_{j=1}^m \int_{-\infty}^\infty \int_0^\infty \Delta_B^{(\gamma+q)/2} K_{j,q}(s, \tau, t) T_{x,y}^{s,\tau} g_j(x, y) \tau^{2\nu+1} ds d\tau. \quad (12)$$

Integrating by parts in formula (12), we obtain, taking into account the commutativity of Δ_B^r and $T_{x,y}^{s,t}$,

$$u(x, y, t) = C \sum_{j=1}^m \int_{-\infty}^{\infty} \int_0^{\infty} K_{j,q}(s, \tau, t) T_{x,y}^{s,t} \Delta_B^{(\gamma+q)/2} g_j(x, y) \tau^{2\nu+1} ds d\tau, \quad (13)$$

where the operator $\Delta_B = \sum \partial^2 / \partial x_i^2 + B_y$.

If $2\nu + 1$ is an integer (and consequently q is an integer), then the associated kernels have the form

$$\begin{aligned} K_{j,q}(x, y, t) &= \int_0^\pi \sin^{2\nu} \varphi \times \\ &\times d\varphi \int_{C_{R,\delta}} d\beta \int_{|\eta|=1} (x\dot{\xi} + y\dot{\sigma} \cos \varphi + t\beta)^{m_j+q} \times \\ &\times \left(\ln \frac{(x\dot{\xi} + y\dot{\sigma} \cos \varphi + t\beta)}{i} + C \right) \frac{N_j}{M^+} \dot{\sigma}^{2\nu+1} d\omega_\eta. \end{aligned} \quad (14)$$

If, however, $2\nu + 1$ is a fractional number (hence q is fractional), then the associated kernels have the form

$$\begin{aligned} K_{j,q}(x, y, t) &= \\ &= C \int_0^\pi \sin^{2\nu} \varphi d\varphi \int_{C_{R,\delta}} d\beta \int_{|\eta|=1} (x\dot{\xi} + y\dot{\sigma} \cos \varphi + t\beta)^{m_j+q} \frac{N_j(\eta, \beta)}{M^+(\eta, \beta)} \dot{\sigma}^{2\nu+1} d\omega_\eta. \end{aligned} \quad (15)$$

Let g_μ be infinitely differentiable finite functions in the half-space E_{n+1} . Using the results of V. I. Kononenko ⁽³⁾ on the expansion of finite functions into weighted plane waves, it is not difficult to show that formula (13) gives a solution of problem (1)–(2). Each of the kernels $K_{j,q}$ is analytic at all points of the half-space $t \geq 0$, with the exception of the origin. As for the origin, it is a pole of a certain order.

Theorem. The kernels $K_{j,q}(x, y, t)$, for $t \geq 0$ and $(x, y) \neq 0$, are infinitely differentiable functions, and for their derivatives the following estimates hold:

1. If $2\nu + 1$ is an integer and $0 \leq s + 2r \leq m_j + q$, then the inequality

$$|D_x^s B_y^r K_{j,q}| \leq C (|x|^2 + y^2 + t^2)^{(m_j+q-s-2r)/2} \times \\ \times \left(1 + \ln (|x|^2 + y^2 + t^2)^{1/2}\right) \quad (16)$$

holds.

If $s + 2r \geq m_j + q + 1$, then $D_x^s B_y^r K_{j,q}$ is homogeneous of degree $m_j + q - s - 2r$, and the logarithmic term in (16) may be omitted.

2. If $2\nu + 1$ is a fractional number (in this case q is also fractional), then

$$|D_x^s B_y^r K_{j,q}| \leq C (|x|^2 + y^2 + t^2)^{(m_j+q-s-2r)/2}. \quad (17)$$

Analogous assertions are also valid for the Poisson kernels K_j .

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³ V. I. Kononenko, DAN, 172, No. 2 (1967).

Note: Figure translations are in progress. See original paper for figures.

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