

Positional operatives

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1968

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Abstract

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UDC 519.48

MATHEMATICS

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ON POSITIONAL OPERATIVES

(Presented by Academician V. M. Glushkov, 23 II 1968)

In the present note simple positional operatives are studied ^(1,2). In particular, the theory of completely simple semigroups of Rees–Sushkevich–Clifford ^(3,4) is generalized to this class of algebras.

1. Let $\mathfrak{P} = \{\rho_k\}_{k=1}^n$ be a family of permutations ρ_k of the set $J_{2n-1} = \{1, 2, \dots, 2n-1\}$, with $\rho_1 = \rho_n = \varepsilon$ (ε is the identity permutation). An operative S is called a $\mathfrak{P}$ -operative if, for any $k \in J_n = \{1, 2, \dots, n\}$, $x_i \in S$,

$$x_1 x_2 \cdots x_{k-1} (x_k x_{k+1} \cdots x_{k+n-1}) x_{k+n} \cdots x_{2n-1} = x_{\rho_k 1} x_{\rho_k 2} \cdots x_{\rho_k (2n-2)} x_{\rho_k (2n-1)}. \quad (1)$$

S is called a π -operative if, for every $k \in J_n$, $\rho_k \{k, k+1, \dots, k+n-1\} = \{k, k+1, \dots, k+n-1\}$. In other words, with a π -operative S one can associate a family $\pi = \{\sigma_k, \pi_k\}_{k=1}^n$ of permutations σ_k, π_k of the set J_n such that $\pi_1 = \pi_n = \sigma_1 = \sigma_n = \varepsilon$, $\sigma_k k = k$, and for any $x_i, y_i \in S$

$$x_1 x_2 \cdots x_{k-1} (y_1 y_2 \cdots y_n) x_{k+1} \cdots x_n = x_{\sigma_k 1} x_{\sigma_k 2} \cdots x_{\sigma_k (k-1)} y_{\pi_k 1} \cdots y_{\pi_k n} x_{\sigma_k (k+1)} \cdots x_{\sigma_k n}.$$

A π -operative is called associative if $\sigma_k = \pi_k = \varepsilon$ for all $k \in J_n$; alternative if n is odd and all $\sigma_k = \pi_{2j+1} = \varepsilon$, $\pi_{2j} = \tau$, where $\tau i = n - i + 1$.

A subset A of an operative S which is a (k) -ideal ^(1,2), for $k = 1, 2, \dots, n$, is called an ideal; for $k = 1$ and $k = n$, a two-sided ideal. An operative S is called i -simple if it contains no ideals distinct from zero and S and is not isomorphic to the operative $V_n = \{a, 0\}$, where 0 is zero and $a^n = 0$.

2. **Theorem 1.** Every i -simple $\mathfrak{P}$ -operative S with zero 0 is the union of its two-sided ideals A_γ : $S = \bigcup_\gamma A_\gamma$, the number of which $|\Gamma|$ is a divisor of the number $n-1$; $A_\gamma \cap A_\delta = \{0\}$ for $\gamma \neq \delta$. If T is an i -simple $\mathfrak{P}$ -operative without zero, then T contains no proper two-sided ideals.

Let a be some fixed element of $S \setminus \{0\}$ and let $a = a_1 a_2 \cdots a_n$, where $a_j \in S$. Denote by A_j that one of the two-sided ideals A_γ which contains the element a_j ; by ξ_{ki} , the least positive residue of the number $\rho_k^{-1}(i + k - 1)$ modulo $n - 1$.

Theorem 2. For any $k, j \in J_{n-1} = \{1, 2, \dots, n - 1\}$,

$$A_k = A_{\xi_{k1}} A_{\xi_{k2}} A_{\xi_{k3}} \cdots A_{\xi_{kn}}, \quad A_{\xi_{k1}} = A_{\xi_{kn}} = A_k,$$

$$A A_{k_2} A_{k_3} \cdots A_{k_n} = \{0\}, \quad \text{if } k_j \neq \xi_{kj}.$$

3. Introduce the following equivalence δ on the set J_{n-1} : $j\delta k$ if and only if $A_j = A_k$.

Theorem 3. $(\xi_j \xi_{ki})\delta(\xi_{\xi_{ki}})$ for any $i, j, k \in J_n$; all equivalence classes of δ consist of one and the same number of elements.

Denote by l the least positive number such that $A_{l+1} = A_1$ and $\pi_{l+1} = \varepsilon$. For associative operatives the equivalence δ turns out to be congruence modulo l . If for an alternative operative δ is not congruence

mod l , then A_i can be chosen so that $j\delta k$ if and only if $j \equiv k \pmod{l}$ or $j + k \equiv 1 \pmod{l}$.

4. The theorem below makes it possible to carry over to π -operatives the theory of ideal factors of semigroups ^(3,4):

Theorem 4. Let M be a minimal ideal* of a π -operative S . Then either $M^n = \{0\}$, or M is an i -simple operative.

5. An i -simple operative S is called j -**simple** if $S^{k-1} a S^{n-k} \neq \{0\}$ for all $a \in S \setminus \{0\}$, $k \in J_n$.

Theorem 5. If S is a j -simple $\mathfrak{p}$ -operative, then all A_γ (see item 2) are its minimal two-sided ideals.

The two-sided ideals A_γ of a j -simple operative S will be called its components. A π -operative S is called a π^* -**operative** if $\pi_k\{1, n\} = \{1, n\}$ for any $k \in J_n$. In particular, an associative and an alternative are π^* -operatives.

Theorem 6. Every i -simple π^* -operative is j -simple.

6. A j -simple operative S is called c -**simple** if each of its components contains a minimal left and a minimal right ideal; a c -simple $\mathfrak{p}$ -operative is called c_1 -**simple** (c_2 -**simple**) if it contains exactly one component (respectively, if each of its left or right ideals is a two-sided ideal).

Theorem 7. In order that an i -simple associative be c -simple, it is necessary and sufficient that it contain a minimal left ideal and a minimal right ideal.

Theorem 8. Every c -simple $\mathfrak{p}$ -operative without zero is isomorphic to the suboperative of all nonzero elements of some c_1 -simple $\mathfrak{p}$ -operative with zero.

Theorem 9. Every c_2 -simple $\mathfrak{p}$ -operative is similar ^(1,2) to a nonzero c_2 -simple π^* -operative in which all $\sigma_k = \varepsilon$.

7. A subset B of an operative S is called its **biideal** if $BS^{2n-3}B \subseteq B$.

Theorem 10. In order that a j -simple $\mathfrak{p}$ -operative be c -simple, it is necessary and sufficient that each of its components contain at least one minimal biideal.

Theorem 11. In order that an i -simple π^* -operative (in particular, an associative or an alternative) be c -simple, it is necessary and sufficient that it contain a minimal biideal.

The last criterion is apparently new even for $n = 2$ (i.e., for semigroups). The theorem below complements one of the results of the articles ^(1,2).

Theorem 12. Every two-sided invertible element of a $\mathfrak{p}$ -operative is (k) -invertible for any $k \in J_n$. In order that all elements of a $\mathfrak{p}$ -operative S be two-sided invertible, it is necessary and sufficient that S have no biideals distinct from the operative S itself.

Theorem 13. In order that an associative S be an n -group, it is necessary and sufficient that it contain no biideals distinct from S .

For $n = 2$ the last theorem is given in ⁽⁴⁾.

8. Let Γ, I, Λ be nonempty sets of indices; $I = \bigcup_{\gamma \in \Gamma} I_\gamma$, $\Lambda = \bigcup_{\gamma \in \Gamma} \Lambda_\gamma$, where the I_γ (respectively, Λ_γ) are pairwise disjoint subsets of the set I (Λ). The set $B = \bigcup_{\gamma \in \Gamma} I_\gamma \times \Lambda_\gamma$ will be called the **quasidiagonal** of the Cartesian product $I \times \Lambda$.

Let, further, $G_0 = G \cup \{0\}$ be a group with an externally adjoined zero ⁽³⁾. On the Cartesian product $G_0 \times B$ define the equivalence

$$\sim: (g; i, \lambda) \approx (g'; i', \lambda')$$

if and only if $(g; i, \lambda) = (g'; i', \lambda')$ or $g = g' = 0$. Denote by $G \circ B$ the quotient set of the Cartesian pro-

* The concepts of a minimal ideal, of a chain, etc. are defined as in the theory of semigroups.

of the product $G_0 \times B$ by the equivalence $\sim$. One may assume that $G \circ B$ consists of all triples $(g; i, \lambda)$, where $g \in G_0$, $(i, \lambda) \in B$, with all triples $0 = (0; i, \lambda)$ identified with one another. If (i, λ) is a fixed element of B' and $F \subseteq G_0$, then by $(F; i, \lambda)$ we denote the set of all elements $(g; i, \lambda)$, where $g \in F$.

It turns out that for an arbitrary c -simple p -operative S with zero there exist a group G and a quasidiagonal B such that $S = G \circ B$. In this case 0 is the zero of the operative S , and the set $(G_0; i, \lambda)$, for any $(i, \lambda) \in B$, is its minimal biideal. If S is c_1 -simple, then $B = I \times \Lambda$.

9. A **matrix** (an ΛI -matrix) over a group with zero G_0 is an arbitrary function P assigning to each pair of elements $\lambda \in \Lambda$, $i \in I$ an element $p_{\lambda i} \in G_0$.

Let an operative S be defined on the set $G \circ B$ (Sec. 8); for any $k = 1, 2, \dots, n$ denote by v_k, v'_k a pair of variables, one of which is equal to $i_k (\in I)$, the other to $\lambda_k (\in \Lambda)$, with (i_k, λ_k) ranging over the whole set B , $v_1 = x_1, v'_n = i_n$. Suppose that for each $k \in J_{n-1}$ a regular $(3, 4)$ -matrix $p_k(v_k, v'_{k+1})$ over the semigroup G_0 is defined, and for any indices $\gamma \in \Gamma$, $k \in J_{n-1}$ an automorphism (or inverse automorphism) $\psi_k^{(\gamma)}$ of the semigroup G_0 . Moreover, if the group G is not abelian and $v'_k = i_k, v_k = \lambda_k$ (respectively $v'_k = \lambda_k, v_k = i_k$), then $\psi_k^{(\gamma)}$, for any $\gamma \in \Gamma$, is an automorphism (respectively an inverse automorphism) of the semigroup G_0 . We shall call S a **Rees operative** over the group G if, for any $(x_j; i_j, \lambda_j) \in S$,

$$(x_1; i_1, \lambda_1)(x_2; i_2, \lambda_2) \dots (x_n; i_n, \lambda_n) = (z; i_1, \lambda_n), \quad (2)$$

where, for $i_1 \in I_\gamma$,

$$z = x_1 \cdot p_1(v_1, v'_2) \cdot \psi_2^{(\gamma)} x_2 \cdot p_2(v_2, v'_3) \dots \psi_{n-1}^{(\gamma)} x_{n-1} \cdot p_{n-1}(v_{n-1}, i_n) \cdot x_n.$$

Theorem 14. *Every c -simple π -operative S is isomorphic to a Rees operative over some group G . If the group G is not abelian, then S is either associative or alternative.*

For c -simple p -operatives Theorem 14 is, generally speaking, not true (cf. Sec. 8 and Theorem 9).

10. Let

$$B = \bigcup_{\gamma=1}^t I_\gamma \times \Lambda_\gamma;$$

G be a group; $P = (p_{\lambda i})$ a regular ΛI -matrix over G_0 for which from $p_{\lambda i} \neq 0$ and $\lambda \in \Lambda_\gamma$ it follows that $i \in I_{\gamma+1}$ (from $\lambda \in \Lambda_t$ it follows that $i \in I_1$); a an element of G ; ψ an automorphism of the semigroup G_0 such that $\psi^t = \lambda_a^{-1}$, where $n-1 = lt$, $\lambda_a x = a^{-1}xa$. Denote

$$j * \gamma = \left[\frac{1}{t}(j + \gamma - 2) \right]$$

(here $[m]$ is the integer part of the number m). Next let $S(G^{(l)}, B, P)$ be the operative defined on the set $G \circ B$ (Secs. 8, 9) with operation (2), where for $i_1 \in I_\gamma$

$$z = \psi^{1*\gamma}(x_1 \cdot p_{x_1 i_2}) \cdot \psi^{2*\gamma}(x_2 \cdot p_{x_2 i_3}) \cdot \dots \cdot \psi^{(n-1)*\gamma}(x_{n-1} \cdot p_{x_{n-1} i_n}) \cdot G \cdot X_n.$$

Theorem 15. $S(G^{(l)}, B, P)$ is a c -simple associative operative with zero. Every c -simple associative operative with zero is isomorphic to some operative $S(G^{(l)}, B, P)$.

From this there easily follows a description of c -simple associatives without zero (they all turn out to be bundles of mutually isomorphic n -groups) and, in particular, of c -simple associatives without proper right ideals ⁽⁵⁾. The author has found all isomorphisms of c -simple associatives.

11. We restrict ourselves here to the description only of c_1 -simple alternatives. Let $n = 2l + 1$; G be a group; $B = I \times \Lambda$; a, s, c be elements of G ; ψ an automorphism; ξ an inverse automorphism of the semigroup G_0 such that

$$\psi^l = \lambda_a^{-1}, \quad \xi^2 = \lambda_c, \quad \xi\psi = \lambda_{asc}\psi^{l-1}\xi, \quad \psi a = a, \quad \xi a = \left\{ \prod_{j=1}^{l-1} \psi^j(sc) \right\} \cdot asc;$$

$P = (p_{\lambda\lambda})$ and $Q = (q_{ij})$ be, respectively, a regular $\Lambda\Lambda$ -matrix and an II -matrix over G_0 , satisfying, for any $i, j \in I$, $\lambda, \lambda \in \Lambda$, the condi-

to the conditions $\xi p_{\lambda\mu} = c^{-1}p_{\chi\lambda}$, $\xi q_{ji} = \psi(s \cdot q_{ij})$. Let, furthermore, $S(G, B, P, Q, \xi)$ be an operative defined on the set $G \circ B$ with action (2), where

$$z = \left\{ \prod_{j=1}^l \psi^{j-1}(x_{2j-1} \cdot p_{x_{2j-1}x_{2j}} \cdot \xi x_{2j} \cdot q_{i_{2j}i_{2j+1}}) \right\} \cdot a \cdot x_n.$$

Theorem 16. $S(G, B, P, Q, \xi)$ is a c_1 -simple alternative. Every c_1 -simple alternative is isomorphic to some operative $S(G, B, P, Q, \xi)$.

For $n = 3$ this result was found in papers ^(6, 7).

12. In describing a π -operative of Rees S over an abelian group G (see Theorem 14), as in Sec. 11, we shall restrict ourselves only to c_1 -simple operatives (after Secs. 10–11 one may assume that S is neither associative nor alternative). For such an operative S , all $\sigma_k = \varepsilon$. The action in S is still determined by formula (2), where

$$\begin{aligned} z &= z' \cdot p_1(x_1, v'_2) \cdot p_2(v_2, v'_3) \cdot \dots \cdot p_{n-1}(v_{n-1}, i_n), \\ z' &= x_1 \cdot \psi_2 x_2 \cdot \psi_3 x_3 \cdot \dots \cdot \psi_{n-1} x_{n-1} \cdot x_n. \end{aligned} \quad (3)$$

Formula (3) defines on the group G the π -commutator ^(1, 2), and thereby singles out all dependencies among the automorphisms ψ_k . All matrices $p_k(v_k, v'_{k+1})$ satisfy one of the following three conditions:

- 1) The matrices p_k do not depend on the indices $\lambda \in \Lambda$ (i.e. have the form $p_k(i_k, i_{k+1})$, in particular $p_k(i_k)$, $p_k(i_{k+1})$, or $p_k = c$.

- 2) The matrices p_k do not depend on the indices $i \in I$.
- 3) $p_1 = p_1(v'_2)$, $p_{n-1} = p_{n-1}(v_{n-1})$; if the matrix p_k depends on i_k (on λ_k), then p_{k-1} and p_{k+1} do not depend on λ_k (respectively on i_k).

Thus the functions $p_1, p_2, \dots, p_{n-1}$ depend not on $2n - 2$ variables i_k, λ_k (see Secs. 9-11), but only on $n - 1$ variables. One can also reduce the number of the functions p_k themselves (as in Secs. 10-11); we shall not do this here.

13. Let S be a c_1 -simple Rees operative (see Secs. 6, 9—in this case $|\Gamma| = 1$); let N be a normal divisor of the group G such that $\psi_{jN} = N$ for every $j \in J_n$ ($\psi_j = \psi_j^{(\gamma)}$); let σ_I be an equivalence on the set I such that, for any matrices p_k, p_l having the form $p_k = p_k(i, v')$ or $p_l = p_l(v, i)$, it follows from $i\sigma_I i'$ that $p_k(i, v') \in p_k(i', v')N$ and $p_l(v, i) \in p_l(v, i')N$; analogously an equivalence σ_Λ is defined on the set Λ . Let, further, $\rho = \rho(N, \sigma_I, \sigma_\Lambda)$ be the following equivalence on the operative S : $(x; i, \lambda)\rho(y; j, \chi)$ if and only if $x = y = o$ or $y \in xN$, $i\sigma_I j$, $\lambda\sigma_\Lambda \chi$.

It follows from Sec. 9 that ρ is a congruence of the operative S .

The converse of this assertion is as follows. If ρ is a congruence on a c_1 -simple π -operative S , then an operative of Rees isomorphic to S can be chosen so that ρ has the form $\rho\{N, \sigma_I, \sigma_\Lambda\}$. We shall not enumerate here the isomorphisms of Rees operatives.

For $n = 2$ (i.e. for semigroups) this result is contained in paper ⁽⁸⁾ (see also ⁽⁴⁾). From this there easily follows a description of the homomorphisms of π -operatives all of whose elements are two-sided invertible ^(1, 2).

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Received
19 II 1968

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